1st International Workshop On High-Order CFD Methods

January 7-8, 2012

at the 50th AIAA Aerospace Sciences Meeting, Nashville, Tennessee

Final abstract for ONERA Taylor-Green DG participation

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Abstract

DG and Fourier pseudo-spectral computations are performed for the Taylor-Green vortex problem at low and moderate Reynolds numbers. The aim is to assess the potential of high-order methods for the direct numerical simulations of turbulent flows. The reference solution provided in the context of the Workshop is given by an over resolved computation carried out using a Fourier spectral code. In particular, we will focus on the evolution of enstrophy and the turbulent kinetic energy decay. Further comparisons are made in physical and spectral space with respect to a reference solution computed employing a pseudo-spectral code.

1 Introduction

AGHORA is an in-house CFD solver developed at ONERA. The full set of compressible Navier-Stokes equations can be solved in two or three dimensions. So far, AGHORA supports tetrahedral or hexaedral mesh elements. The discretization scheme is based on a discontinuous Galerkin modal approach, the order of the approximation depending directly on the specified polynomial degree.

The test case considered in the context of the workshop is the direct numerical simulation (DNS) of the Taylor-Green vortex flow. This case is a critical test for numerical schemes, as the convective and viscous terms play important roles. The convective terms generate the energy cascade between the different scales of the turbulent flow, and accurate representation of the viscous flux is needed because it has to dissipate the energy around the highest wavenumbers.

We have performed DG computations for low and moderate Reynolds numbers, respectively $Re = 280$ and $Re = 1600$. The reference solutions are provided in the context of the workshop (statiscal properties of the flow and magnitude of the vorticity field are available). We have also performed our own pseudo-spectral computations for more in-depth comparisons of the spectral and physical characteristics of the flow.

2 Case details

In order to assess the DG discretization for direct numerical simulation, the Taylor-Green vortex is a case of interest, since it allows the analytic initialization of a two direction isotropic field ($u_0^{rms} = v_0^{rms}, w_0^{rms} = 0$) in physical space [1]. It corresponds in spectral space to an injection of energy at the wavenumber $k_i = \frac{2}{L}$, $2\pi L$ being the box size. The initial field is given by the following functions :

$$\begin{aligned} u_0(x,y,z) &= V_0 sin\left(\frac{x}{L}\right) cos\left(\frac{y}{L}\right) cos\left(\frac{z}{L}\right) \\ v_0(x,y,z) &= -V_0 cos\left(\frac{x}{L}\right) sin\left(\frac{y}{L}\right) cos\left(\frac{z}{L}\right) \\ w_0(x,y,z) &= 0 \end{aligned}$$

Although this case is meant to be incompressible, one can extend the initial profile to compressible equations, imposing the following initial pressure field :

$$p_0(x,y,z) = p_\infty + \frac{\rho_0}{16}\left(cos(\frac{2x}{L}) + cos(\frac{2x}{L})\right)\left(cos(\frac{2z}{L}) + 2\right)$$

The incompressible limit is reached by setting the Mach number for all simulation to $M = 0.1$. All the quantities involved in the study are nondimensionalised using L, V_0 and ρ_0.

The DG computations are performed on meshes containing cubic elements. The number of degrees of freedom (DOFs) is defined as the product of the total elements number times the dimension of the polynomial basis ($(p+1)^3$ in the case of cubic elements, where p is the polynomial order). We also use a de-aliased Fourier pseudo-spectral code developped at Cambridge University named Fergus [2] to obtain a reference solution in each case.

In order to analyse the results, we define space averaged quantities related to the turbulent motion over a volume V:

- The mean turbulent kinetic energy E_k (also named TKE) :

$$E_k = \frac{1}{V}\int_V \frac{\mathbf{u}.\mathbf{u}}{2} d\mathbf{x}$$

- The mean enstrophy Ω:

$$\Omega = \frac{1}{V}\int_V \frac{\omega.\omega}{2} d\mathbf{x}$$

- The mean kinetic energy dissipation rate ε, which can be evaluated in two ways :

$$\varepsilon = -\frac{dE_k}{dt} = \frac{2\mu}{V}\int_V \frac{\mathbf{S}:\mathbf{S}}{2} d\mathbf{x}$$

- The incompressible and compressible contributions to the mean dissipation rate, ε_1 and ε_3 respectively:

$$\varepsilon_1 = \frac{2\mu}{V}\int_V \mathbf{S}^d : \mathbf{S}^d d\mathbf{x}\ ,\ \varepsilon_3 = \frac{1}{V}\int_V p\nabla.\mathbf{u} dV$$

  where $\mathbf{S}^d$ is the deviatoric part of the strain rate tensor, $\mathbf{u}$ the velocity vector and ω the vorticity vector.

The temporal evolution of these mean quantities allows us to monitor the quality of the solution over time. Especially, the mean enstrophy and dissipation rates are representative of the accurate discretization of the gradients.

For local comparisons in physical space, we can plot the vorticity modulus or the values of Q criterion, which is related to the stretched vortex tubes appearing when the turbulence is developped in the flow. The best way to visualize these structures is to plot positives values of Q, i.e. :

$$Q = \frac{1}{2}\left(\mathbf{R} : \mathbf{R} - \mathbf{S} : \mathbf{S}\right) > 0$$

where $\mathbf{R}$ is the rotationnal tensor and $\mathbf{S}$ the strain rate tensor.

3 Numerical method

We briefly describe here the main features of the DG method used to compute the Taylor-Green vortex case. The solution in each element is expressed in terms of a polynomial expansion. The basis functions in two or three dimensions are built using tensor products of 1D polynomials. In the particular case of parallelepipeds, the Legendre polynomial basis is used. For tetrahedral elements, we use the orthogonal Dubiner basis, based on Jacobi polynomials. A modified Gram-Schmidt orthonormalization procedure may also be used for general-shaped elements. We employ over integration for the de-alising of the method. This implies the definition of $(p+1)^3$ quadrature points on each parallelepipedic element. The Lax-Friedrichs flux is chosen to approximate the convective fluxes across the interface of elements. The BR2 scheme [3, 4] is used to discretize the viscous fluxes, both in cells and on faces using local and global lifting operators. Finally, an explicit third order low storage Runge-Kutta scheme is used for time stepping. The approximation order of the spatial scheme can be chosen by specifying the polynomial degree.

4 Computational details

The domain consists of a cubic box with periodic boundary conditions on all faces. The length of the domain is 2π in each of the axis directions. The meshes considered in this study are cartesian with regular grid spacing, therefore composed of cubic elements.

Two DG and one pseudo-spectral (PS) computations have been performed for both the low and moderate Reynolds number cases. For the latter, parallel runs have been performed. Tab. 1 summarizes the details of the 6 computations carried out.

Method	Case	Accuracy (p+1)	Nb. cores	Nb. DOFs/Elements	Δt
DG	$Re = 280$	3	1	$96^3/\ 32^3$	1.45×10^{-3}
DG	$Re = 280$	4	1	$96^3/\ 24^3$	1.95×10^{-3}
PS	$Re = 280$	-	1	96^3	6.28×10^{-3}
DG	$Re = 1600$	4	256	$256^3/\ 64^3$	7.29×10^{-4}
DG	$Re = 1600$	4	432	$384^3/\ 96^3$	4.86×10^{-4}
PS	$Re = 1600$	-	1	384^3	3.14×10^{-3}

Table 1 : Detail of the computations performed

5 Results

5.1 DNS computations at $Re = 280$

We first consider a case with $Re = 280$. At this low Reynolds number, 64^3 Fourier modes are sufficient to represent accurately all the scales of motion, in the limit of $\Delta x = 2\eta$,where η is the Kolmogorov scale, as proposed by Pope [5]. We also performed a computation using 128^3 Fourier modes using the Fergus pseudo-spectral code [2] verifying $\Delta x < \eta$, in order to obtain a reference solution. Two DG computations involving 96^3 degrees of freedom have been carried out using a third and a fourth order-discretization. We first compare the evolution of the mean turbulent kinetic energy, enstrophy and dissipation rate in Fig. 1 to check the capacity of DG schemes to capture the basic dynamics of turbulent flows.

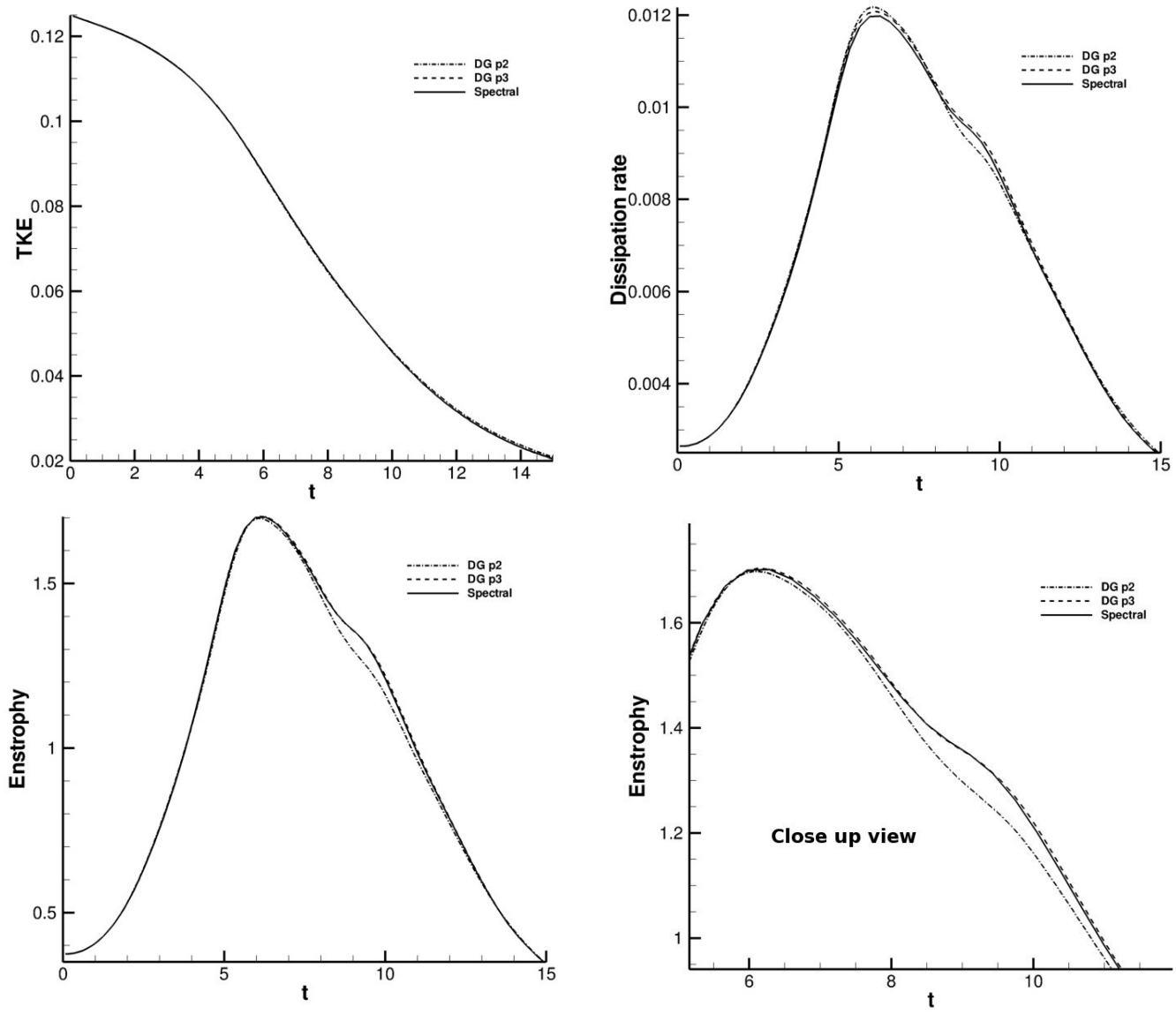

Figure 1: Evolution of mean TKE, dissipation rate and enstrophy evolution for the Taylor-Green simulations at Re=280. Comparison between the DG discretizations and the Fourier pseudo-spectral method.

The differences between the DG and the spectral code solutions are very small for the TKE evolution. The enstrophy decays too abruptly for the DG third-order case, while the fourth-order case shows a good correlation with the spectral result. We observe the same behaviour for the dissipation rate.

In Fig.2, we plot the energy spectra for each solution at a time past the enstrophy peak, for which all scales down to η are well developped in the flow.

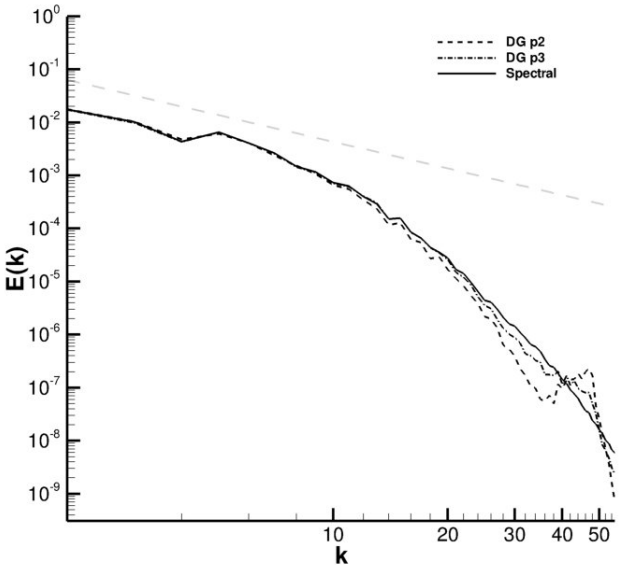

Figure 2: TKE spectra at $t = 9.4$, dashed grey line represents the -5/3 slope

We observe that the DG discretizations act here as a low-pass filter on the turbulent field. This effect is in particular visible for the third-order DG discretization for which the energy content of the smaller scales is not accurately represented. The fourth-order DG case captures more accurately the small-scale phenomena, although an energy damping is still present in the upper range of the spectrum.

5.2 DNS computations at $Re = 1600$

This case requires a greater number of degrees of freedom than for the $Re = 280$ case for the represention of the whole spectrum of turbulent scales. All scales are captured with 512^3 degrees of freedom in the spectral computation although in practice 256^3 degrees of freedom are sufficient to represent the main characteristics of the flow (see van Rees [6]). We performed two computations with the DG code involving 256^3 and 384^3 DOFs. The last one matches the Fourier modes needed for a pseudo-spectral 256^3 computation (as the dealiasing is realised by using $\left(\frac{3}{2} \times 256\right)^3$ Fourier modes in pseudo-spectral methods). Fig. 3 represents the evolution of the turbulent kinetic energy, the dissipation and the enstrophy for both simulations.

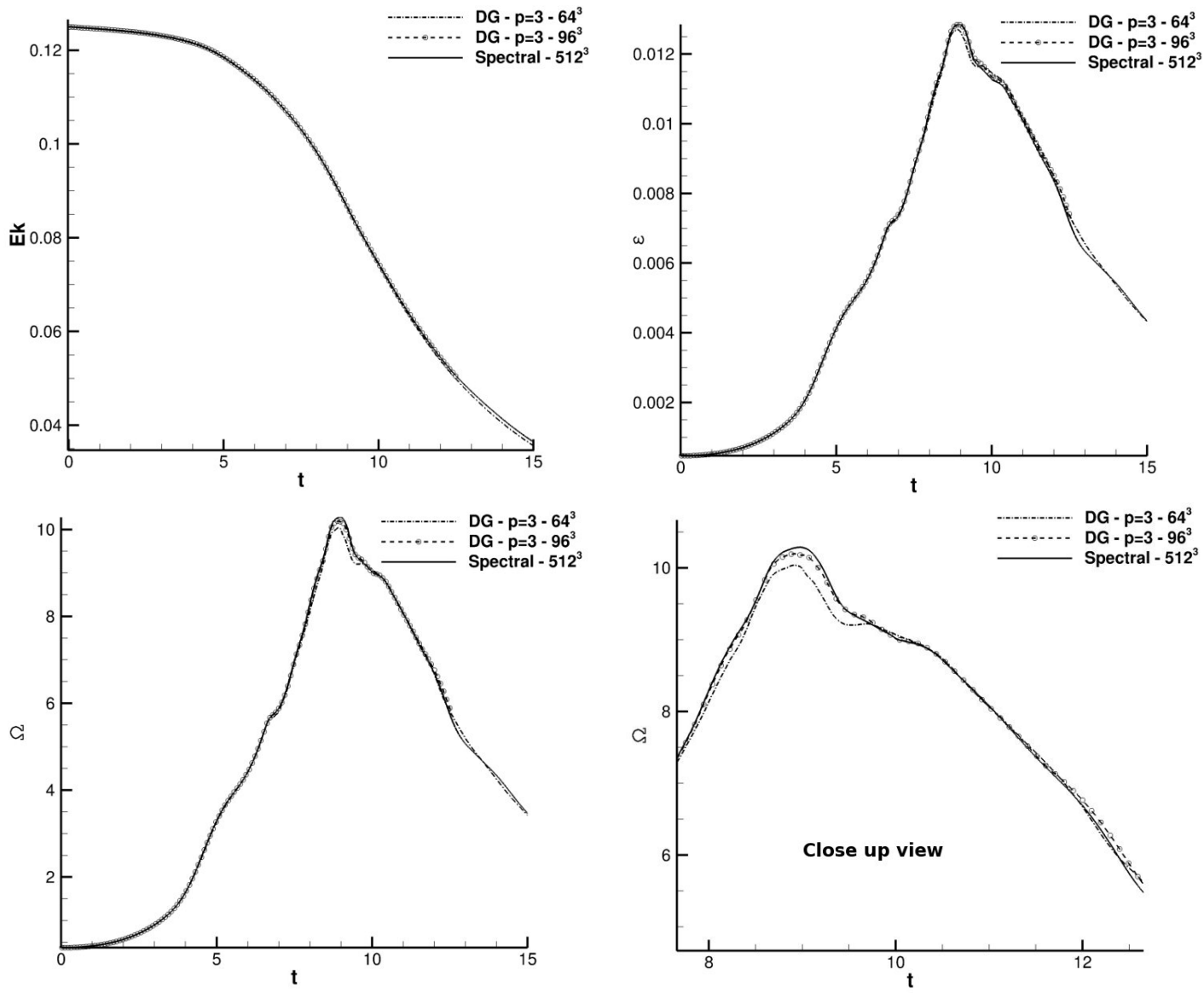

Figure 3: Evolution of mean TKE, dissipation rate and enstrophy evolution for the Taylor-Green simulations at Re=1600. Comparison between the DG discretizations and the Fourier pseudo-spectral method.

Both computations show very good agreement with the spectral reference solution, and in particular during the first phase of the simulation for which the convective phenomena are predominant. Small discrepancies appear, however, near the enstrophy peak, when all the scales of turbulent motion are developped and contribute to the overall dynamics of the flow. To further investigate the cause of these differences, Fig. 4 shows the spectra for a time past the enstrophy peak, for both the DG computation and the pseudo-spectral computation involving 256^3 Fourier modes.

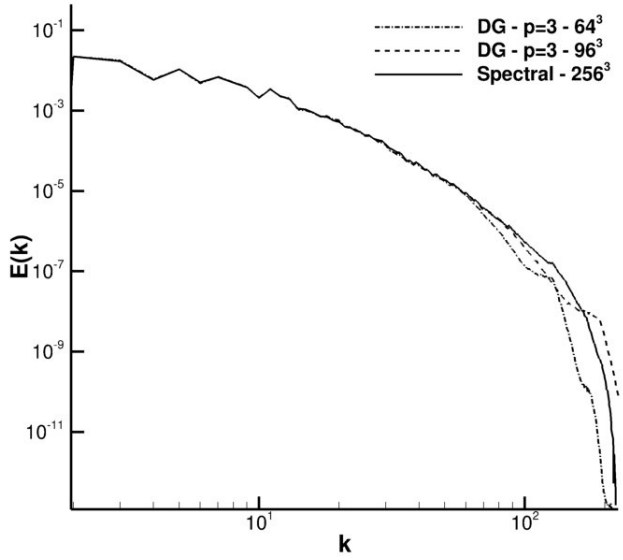

Figure 4: TKE spectra at $t = 8.2$ for the $Re = 1600$ case

We see that the DG computation with more DOFs has a richer spectral content than the under-resolved one. It could be interesting to investigate wether the quality of the spectral representation is improved by performing computations with the number of DOFs but with an increased approximation order.

As the AGHORA code solves the compressible set of Navier-Stokes equations, we can investigate the effects of compressibility if any, by first comparing the values of dissipation given by $\varepsilon = -\frac{dE_k}{dt}$ and $\varepsilon_1 = \frac{2\mu}{V}\int_V \mathbf{S}^d : \mathbf{S}^d d\mathbf{x}$ in Fig.5 and also by plotting ε_3 in Fig. 6 for both DG discretizations.

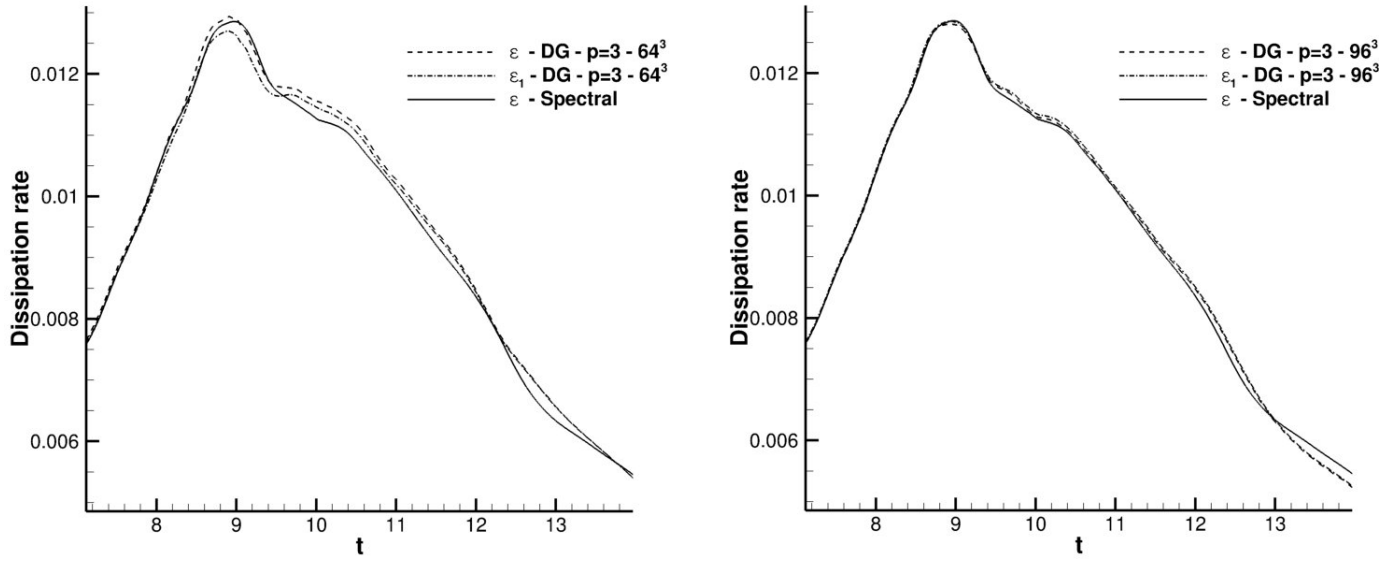

Figure 5: Comparison between ε and ε_1 for the two DG computations

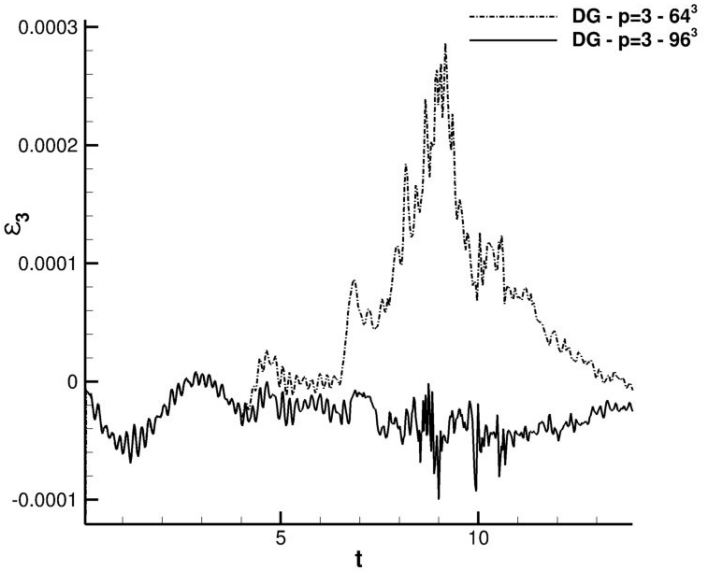

Figure 6: Evolution of ε_3 for the two DG computations

The evolution of dissipation in Fig. 5 shows that the difference between the total dissipation and the incompressible component of the dissipation is significant for the DG 64^3 computation. For the DG 96^3 computation, this difference is almost imperceptible. It therefore seems that compressibility effects play a role in the under-resolved computation. This is confirmed in Fig. 6 which shows that the compressible part of the dissipation is more important for the DG 64^3 computation. The oscillations of ε_3 are probably a consequence of acoustic waves propagating in the flow, as this term is a function of the pressure.

We have also checked the solutions obtained with the DG and spectral computations in physical space. First, we have analysed the field of vorticity magnitude on a quarter of a slice at $x = 0$ in the domain. This is presented in Fig. 7, for both the DG solutions and the data available from van Rees et al. [6].

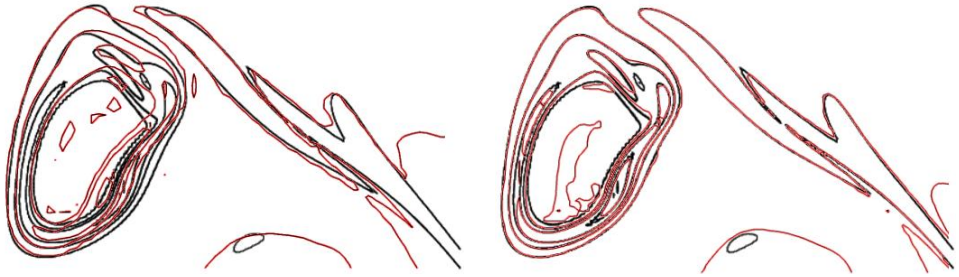

Figure 7: Modulus of vorticity on a slice at $x = 0$ for 512^3 spectral (black), DG 64^3 $p3$ (left) and DG 96^3 $p3$ (right) computations (superimposed in red)

The vorticity map for the under-resolved DG computation (DG 64^3 $p3$) fails to reproduce some of the details and shows some discrepancies with respect to the spectral solution. However, the main structures are relatively similar. The 96^3 DG computation shows a far better agreement. The small differences are observed for low values of the vorticity magnitude (e.g. in the representation of the vortex cores).

To further assess the capacity of the DG discretization to represent the vortical structures of the flow, we have studied the solution based on the Q criterion. This criterion is related to the elongated worm-like vortices which are typical of isotropic turbulent flows. Here we choose the time $t = 10.8$, which corresponds to a time past the enstrophy peak.

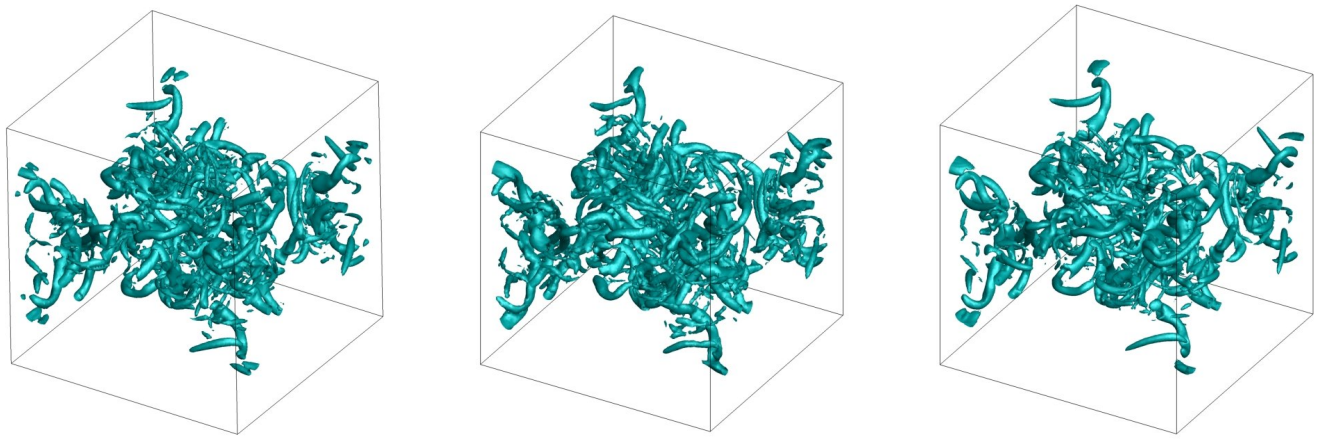

Figure 8: Representation of the Q criterion for 256^3 spectral (left), DG 64^3 $p3$ (center) and DG 96^3 $p3$ (right) computations

Despite the fact that the marginally resolved DG computation provides a poorer representation of the vortex field as seen in Fig. 8, we have found a very good agreement between the three computations in terms of the organisation of turbulent structures in physical space.

Conclusion

We have assessed ONERA's Aghora DG code based on the Taylor-Green vortex flow at $Re = 280$ and $Re = 1600$. The low Reynolds number tests have shown that high-order methods are relevant for unsteady computations of turbulent flows. Indeed, the analysis of the TKE spectra shows that a better representation of the flow is obtained when the order of approximation is increased, at equal number of DOFs. We have also obtained very good agreement between the DG and the Fourier spectral reference computations in terms of the temporal evolution of mean quantities like the turbulent kinetic energy or the enstrophy. This good correlation is also verified for the moderate Reynolds number computations. The results for this case also show the capacity of DG schemes to represent accurately the flow motion in physical space (turbulent structures by looking at iso-Q criterion surfaces and vorticity modulus). However, it seems that resolutions higher than those considered here are required to represent the full spectral content of the flow. It could be interesting to study wether increasing the discretization order while keeping the same number of DOFs is sufficient to capture the structures down to the Kolmogorov scales. The comparison between the dissipation rate computed from the turbulent kinetic energy and from the incompressible part of the strain rate tensor shows that the DG computation with fewer DOFs is more sensitive to compressibility effects. This was verified by analysing the evolution of the compressible part of the dissipation rate, since its value is greater in the under-resolved DG computation. Further investigations are required to find out the origin of this phenomenon.

References

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