Numerical Simulation of the Taylor-Green Vortex at Re=1600 with the Discontinuous Galerkin Spectral Element Method for well-resolved and underresolved scenarios

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Extended Abstract

In the following work, we present the results of selected simulations of the classical Taylor-Green vortex problem with a variant of the Discontinuous Galerkin method (DG) labeled the "Discontinuous Galerking Spectral Element Method" (DGSEM). We consider both the well-resolved (DNS-like) case and the underresolved case and show that stabilized high-order schemes are well suited for this type of simulation and outperform their low-order counterparts.

Taylor-Green Vortex flow

The Taylor-Green vortex flow problem constitutes the simplest flow for which a turbulent energy cascade can be observed numerically. Starting from an initial analytical solution containing only a single length scale, the flow field undergoes a rapid build-up of a fully turbulent dissipative spectrum because of non-linear interactions of the developing eddies (Fig. 1). The resulting flow field exhibits the features of an isotropic, homogeneous turbulence and is often used in code validation or evaluation of numerical approaches to subgrid scale modeling [2], [3], [4].
All our computations were run on a structured Cartesian grid of hexahedral elements, covering a triple-periodic box of size $[-\pi, \pi]^3$. The physical time frame from $0s$ to $20s$ was covered according to the problem description, starting from the initial analytical solution with given velocity and pressure fields, a constant temperature and an essentially incompressible flow field with a Mach number of $Ma = 0.1$.

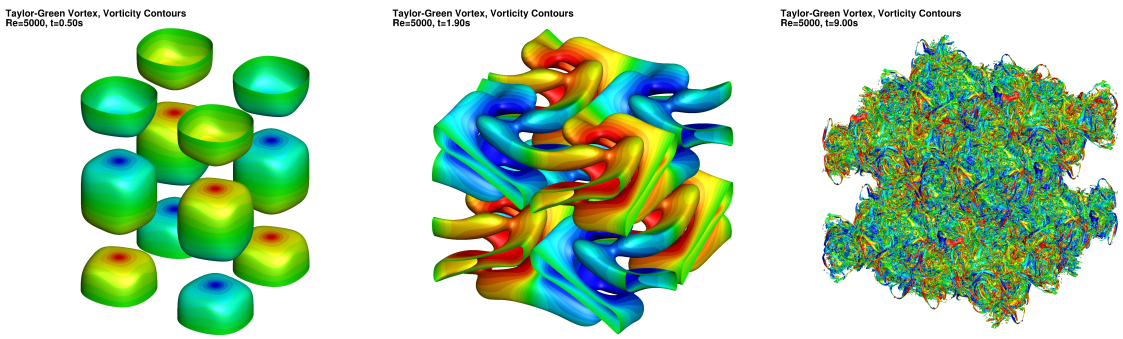

Figure 1: Taylor-Green Vortex ($Re = 5000$). Isocontours of vorticity magnitude, colored by helicity at $t = 0.5s, 1.9s$ and $9.0s$

We have studied the Taylor-Green vortex problem extensively for a range of Reynolds numbers from

$Re = 200$ up to $Re = 5000$ for both resolved and underresolved scenarios. In this work, we will present our findings for two scenarios: a) high-resolution computations of this flow for $Re = 1600$ for a resolution of 256^3 DOF and varying distribution of elements and subcell resolution through polynomial order and b) underresolved simulations of the same flow with only 64^3 DOF.

Code Framework

Our code framework is based on a collocation type formulation of the Discontinuous Galerkin method labeled the "Discontinuous Galerkin Spectral Element method", see Kopriva [7], and solves the compressible Navier-Stokes equations. The implementation allows the selection of arbitrary polynomial order and thus enables us to study the features of high order formulations very efficiently within our framework. Explicit time integration is achieved by a 5-stage 4th order Runge-Kutta scheme.
The code is accompanied by a postprocessing tool for visualization and a-posteriori extraction of relevant flow features and a 3D Fast Fourier transform for the analysis of flow spectra. The whole framework is fully MPI-parallelized, where special care has been taken to achieve a high parallel efficiency and excellent scaling. On the Jugene (IBM BlueGene/P system, Jülich Supercomputing Center) system, a strong scaling of close to 90% was measured on up to 125000 processors [1].
In this work, we present computations performed on the NEC Nehalem cluster (TauBench of 7.6s) and on the Cray XE6 Hermit cluster (TauBench of 15.1s) at the High Performance Computing Center Stuttgart (HLRS) on 128 to 512 cores.

Results for the well-resolved case: 256^3 DOF

As indicated in the test case 3.5 setup description, a resolution of 256^3 DOF is expected to resolve almost all of the flow scales for a Reynolds number of 1600 and is thus very close to a DNS. We have conducted a series of simulations of this test case with varying number of elements and associated polynomial degree, resulting in $\approx 256^3$ DOF for all cases. Table 1 summarizes some selected setups and gives their computational effort in TauBench workunits. Note that the computation with $N = 15$ needed a weak stabilization by filtering, due to the interaction of two effects: Firstly, the resolution of 256^3 DOF with a grid Nyquist wavenumber of $k_{Ny} = 128$ is not sufficient for a full DNS, i.e. parts of the dissipation range cannot by captured on this grid. Secondly, the reduced dissipation of the very high order formulation reduces its tolerance of computational crimes like aliasing errors introduced by the insufficient integration precision of the flux terms [5].

No. of Elements	N	DOF per dir	Stabilization	No. of cores	TAU Work Units
128	1	256	-	128 (Cray XE6)	840,000
64	3	256	-	256 (Cray XE6)	915,000
32	7	256	-	256 (Cray XE6)	929,000
25	9	250	-	125 (Cray XE6)	944,000
21	11	252	-	343 (Cray XE6)	1,670,000
16	15	256	weak	256 (Cray XE6)	2,310,000
64	7	512	-	512 (NEC Nehalem)	38,100,000
24	15	384	-	512 (NEC Nehalem)	14,000,000

Table 1: Selected Taylor-Green vortex computations

Figure 2 shows the results for the kinetic energy dissipation rate over simulated time for the highlighted combinations of h- and p-resolution in table 1, and a zoom-in on the time with a strong dominance of the small scales. As evident from these plots, the high-order simulations with their lower numerical errors outperfom their lower-order counterpart. In particular, the results for $N = 7$ (32x8) are very close to the DNS results.

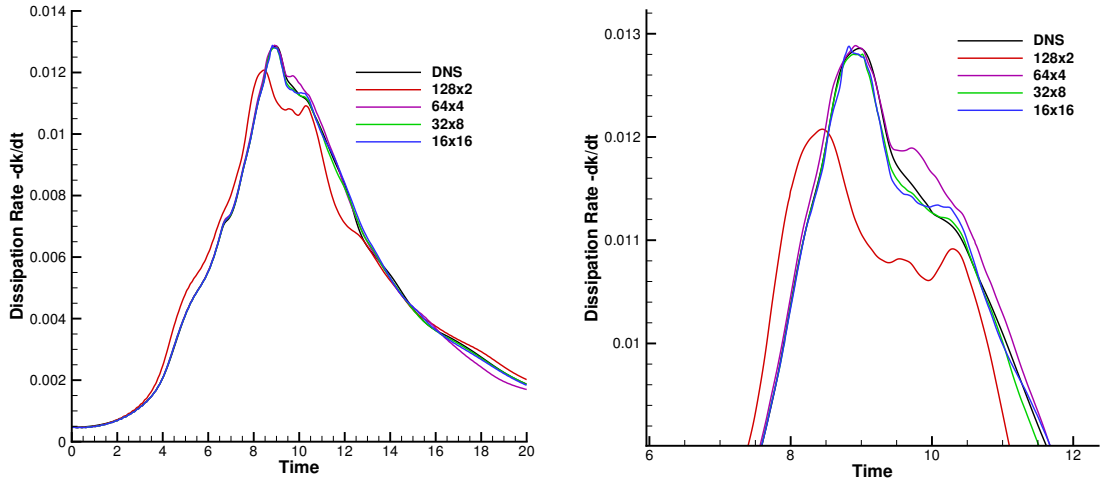

Figure 2: Kinetic energy dissipation rate and zoom in on maximum region

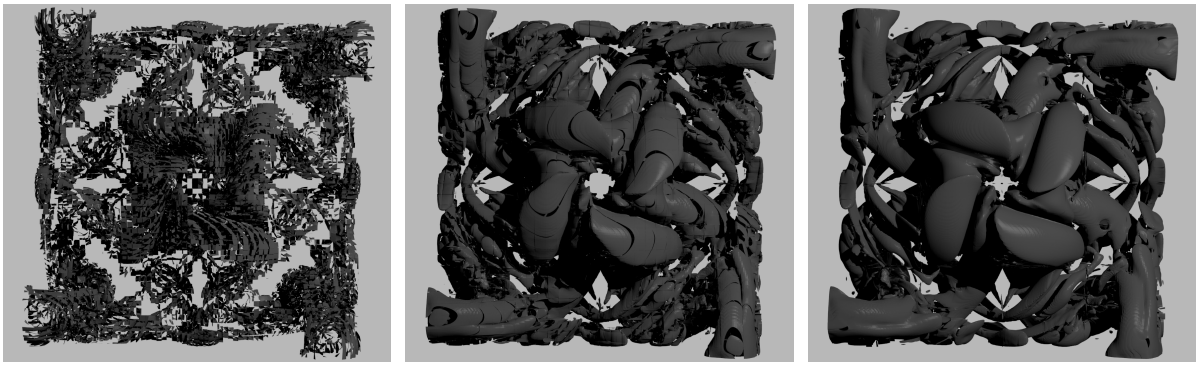

Figure 3: Visualization of vortex detection criterion $\lambda_2 = -1.5$ for $N = 1$, $N = 3$ and $N = 15$ case (left to right, 256^3 DOF in each case)

Figure 3 gives a visual impression of the solution quality for the 256^3 DOF computations by depicting the vortex structure at $t = 8s$. The linear approximation of the solution in each cell ($N = 1$; left plot) captures only the very large structures in the flow and shows strong discontinuities at the grid cell interfaces. Increasing the polynomial order to $N = 3$ (middle plot) results in a significant improvement, large scale structures become considerably smoother and small scale features start to appear, although contaminated by noise. For the very high order computation ($N = 15$, right plot), the clutter is almost gone, small structures are well resolved and the vortex representation is smooth.

Results for the underresolved case: 64^3 DOF

As presented in the previous section, it is obvious that for well-resolved multiscale flows, high order schemes benefit from their superior dissipation and dispersion qualities and outperform low-order formulations. However, for most pratical flow problems, the high Reynolds numbers make a high-resolution simulation prohibitively expensive. In these underresolved cases, the theoretical order of convergence associated with the polynomial approximation as the grid size tends to zero loses relevance, since h is "large" and far from zero. Instead, the dissipative and dispersive error behavior for underresolved wavelengths dominates the approximation quality and is a better measure for the accuracy of the method.
In figure 4, we consider again the Taylor-Green vortex at $Re = 1600$ for a low and a high order approximation with equal nominal resolution, but we reduce the degrees of freedom consecutively by a factor of 2. With decreasing resolution, the numerical error dominates the behavior of the low-order scheme and effectively masks the underlying physics and introduces a much lower "numerical" Reynolds number.

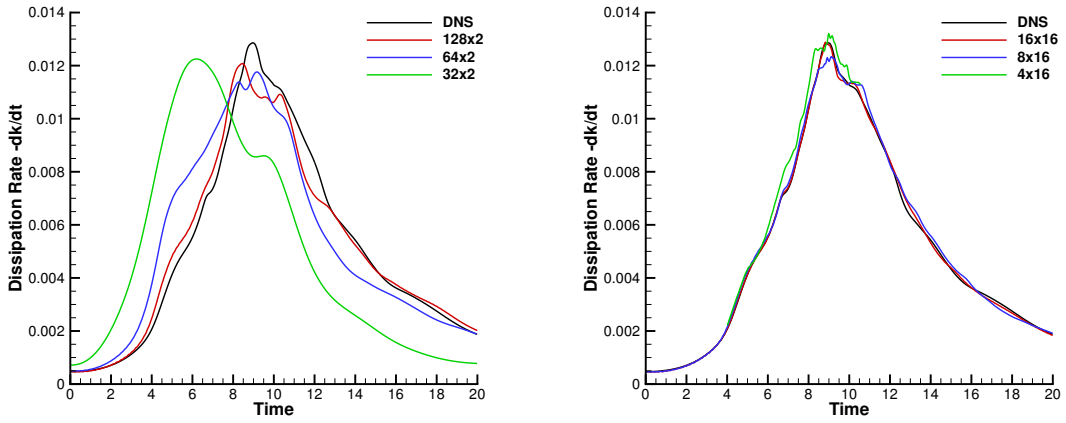

Figure 4: Kinetic energy dissipation rate for reduced resolutions: *left:* $N = 1$, *right:* $N = 15$, curves of the same color have the same no. of total DOF (red - 256^3; blue - 128^3; green - 64^3)

(The maximum of the dissipation rate moves to earlier times, which is characteristic of the Taylor-Green vortex at lower Reynolds numbers, see e.g. [3].) The high-order scheme proves to be more resilient to the reduced number of degrees of freedom and still captures the relevant flow structures satisfactorily, even for the coarse 64^3 DOF resolution. It should be noted however that due to the inherently lower dissipative error and higher suspectibility to aliasing errors, the high order discretizations require a stabilizing mechanism such as overintegration or filtering. These stabilizing techniques and the accuracy of high-order discretizations for underresolved Taylor-Green vortex simulations were investigated in detail in [6].
Figure 5 corroborates these findings for the increasing Reynolds number by comparing the 16^{th} order stabilized computation with 64^3 DOF with state-of-the-art explicit and implict LES formulations with the same resolution. Details can be found again in [6].

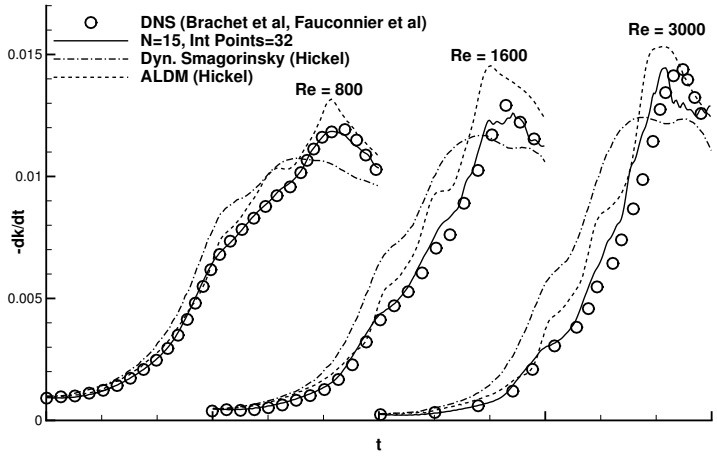

Figure 5: Plot for the kinetic energy dissipation rate for $Re = 800, 1600, 3000$ for $N = 15$ computations compared to DNS and LES reference data. Published in [6].

Conclusion

We have investigated the Taylor-Green vortex flow for well-resolved and underresolved scenarios with the Discontinuous Galerkin Spectral Element method. Our framework is capable of delivering high-order accurate results in a highly efficient way. Investigations into the behavior of underresolved high-order discretizations indicates their potential usefulness for coarse-scale simulations of multi-scale problems.

References

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