Problem C3.5 Direct Numerical Simulation of the Taylor-Green Vortex at $Re = 1600$

1 Overview

This problem is aimed at testing the accuracy and the performance of high-order methods on the direct numerical simulation of a three-dimensional periodic and transitional flow defined by a simple initial condition: the Taylor-Green vortex. The initial flow field is given by

$$
\begin{aligned}
u &= V_0 \sin\left(\frac{x}{L}\right) \cos\left(\frac{y}{L}\right) \cos\left(\frac{z}{L}\right) , \\
v &= -V_0 \cos\left(\frac{x}{L}\right) \sin\left(\frac{y}{L}\right) \cos\left(\frac{z}{L}\right) , \\
w &= 0 , \\
p &= p_0 + \frac{\rho_0 V_0^2}{16} \left(\cos\left(\frac{2x}{L}\right) + \cos\left(\frac{2y}{L}\right) \right) \left(\cos\left(\frac{2z}{L}\right) + 2 \right) .
\end{aligned}
$$

This flow transitions to turbulence, with the creation of small scales, followed by a decay phase similar to decaying homogeneous turbulence (yet here non isotropic), see figure 1.

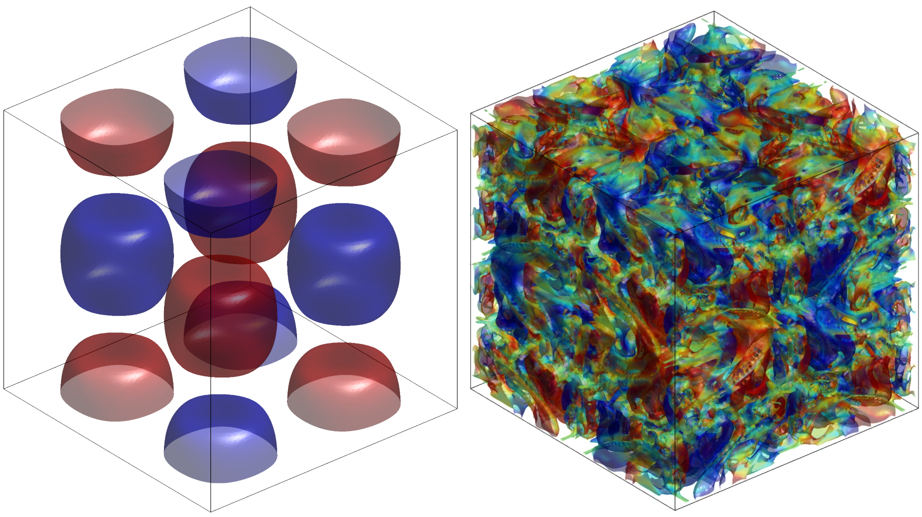

Figure 1: Illustration of Taylor-Green vortex at $t = 0$ (left) and at $t_{\text{final}} = 20\, t_c$ (right): iso-surfaces of the z-component of the dimensionless vorticity.

2 Governing Equations

The flow is governed by the 3D incompressible (i.e., $\rho = \rho_0$) Navier-Stokes equations with constant physical properties. Then, one also does not need to compute the temperature field as the temperature field plays no role in the fluid dynamics.

Alternatively, the flow is governed by the 3D compressible Navier-Stokes equations with constant physical properties and at low Mach number.

3 Flow Conditions

The Reynolds number of the flow is here defined as $Re = \frac{\rho_0\, V_0\, L}{\mu}$ and is equal to 1600.

In case one assumes a compressible flow: the fluid is then a perfect gas with $\gamma = c_p/c_v = 1.4$ and the Prandtl number is $Pr = \frac{\mu\, c_p}{\kappa} = 0.71$, where c_p and c_v are the heat capacities at constant pressure and volume respectively, μ is the dynamic shear viscosity and κ is the heat conductivity. It is also assumed that the gas has zero bulk viscosity: $\mu_v = 0$. The Mach number used is small enough that the solutions obtained for the velocity and pressure fields are indeed very close to those obtained assuming an incompressible flow: $M_0 = \frac{V_0}{c_0} = 0.10$, where c_0 is the speed of sound corresponding to the temperature $T_0 = \frac{p_0}{R\, \rho_0}$. The initial temperature field is taken uniform: $T = T_0$; thus, the initial density field is taken as $\rho = \frac{p}{R\, T_0}$.

The physical duration of the computation is based on the characteristic convective time $t_c = \frac{L}{V_0}$ and is set to $t_{\text{final}} = 20\, t_c$. As the maximum of the dissipation (and thus the smallest turbulent structures) occurs at $t \approx 8\, t_c$, participants can also decide to only compute the flow up to $t = 10\, t_c$ and report solely on those results.

4 Geometry

The flow is computed within a periodic square box defined as $-\pi L \leq x, y, z \leq \pi L$.

5 Boundary Conditions

No boundary conditions required as the domain is periodic in the three directions.

6 Grids

The baseline grid shall contain enough (hexahedral) elements such that approximately 256^3 DOFs (degrees of freedom) are obtained: *e.g.*, 64^3 elements when using $p = 4$ order interpolants for the continuous Galerkin (CG) and/or discontinuous Galerkin (DG) methods. Participants are encouraged, as far as can be afforded, to perform a grid or order convergence study.

7 Mandatory results

Each participant should provide the following outputs:

- The temporal evolution of the kinetic energy integrated on the domain Ω:

$$E_k = \frac{1}{\rho_0\,\Omega}\int_\Omega \rho\,\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\;d\Omega\;.$$

- The temporal evolution of the kinetic energy dissipation rate: $\epsilon = -\frac{dE_k}{dt}$.

  The typical evolution of the dissipation rate is illustrated in figure 2.

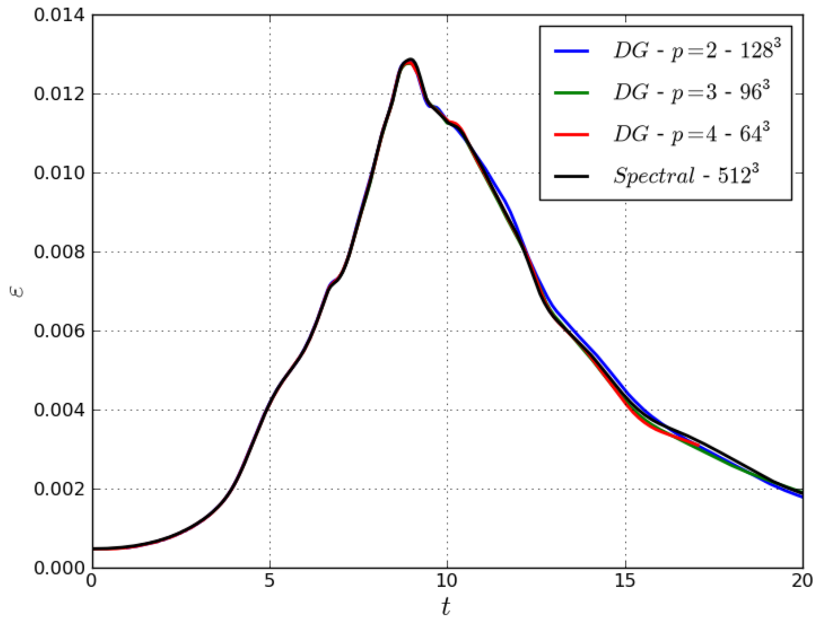

Figure 2: Evolution of the dimensionless energy dissipation rate as a function of the dimensionless time: results of pseudo-spectral code and of variants of a DG code.

- The temporal evolution of the enstrophy integrated on the domain Ω:

$$\mathcal{E} = \frac{1}{\rho_0\,\Omega}\int_\Omega \rho\,\frac{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}{2}\;d\Omega\;.$$

  This is indeed an important diagnostic as ϵ is also exactly equal to $2\frac{\mu}{\rho_0}\mathcal{E}$ for incompressible flow and approximately for compressible flow at low Mach number (cfr. section 8).

Figure 3: Iso-contours of the dimensionless vorticity norm, $\frac{L}{V_0}|\boldsymbol{\omega}| = 1, 5, 10, 20, 30$, on a subset of the periodic face $\frac{x}{L} = -\pi$ at time $\frac{t}{t_c} = 8$. Comparison between the results obtained using the pseudo-spectral code (black) and those obtained using a DG code with $p = 3$ and on a 96^3 mesh (red).

- The vorticity norm on the periodic face $\frac{x}{L} = -\pi$ at time $\frac{t}{t_c} = 8$. An illustration is given in figure 3.

Important: All provided values should be properly non-dimensionalised: e.g., divide t by $\frac{L}{V_0} = t_c$, E_k by V_0^2, ϵ by $\frac{V_0^3}{L} = \frac{V_0^2}{t_c}$, $\mathcal{E}$ by $\frac{V_0^2}{L^2} = \frac{1}{t_c^2}$, etc.

8 Suggested additional results

Furthermore participants are encouraged to provide additional information.

- In incompressible flow ($\rho = \rho_0$), the kinetic energy dissipation rate ϵ, as obtained from the Navier-Stokes equations, is:

$$\epsilon = -\frac{dE_k}{dt} = 2\frac{\mu}{\rho_0}\frac{1}{\Omega}\int_\Omega \boldsymbol{S} : \boldsymbol{S}\; d\Omega$$

  where $\boldsymbol{S}$ is the strain rate tensor. It is also easily verified that this is equal to

$$\epsilon = 2\frac{\mu}{\rho_0}\mathcal{E}\ .$$

  If possible, the temporal evolution of the integral $\frac{1}{\Omega}\int_\Omega \boldsymbol{S} : \boldsymbol{S}\; d\Omega$ shall be reported in addition to $-\frac{dE_k}{dt}$ and the enstrophy.

- In compressible flow, the kinetic energy dissipation rate obtained from the Navier-Stokes equations is the sum of three contributions:

$$\epsilon_1 = 2\frac{\mu}{\rho_0}\frac{1}{\Omega}\int_\Omega \boldsymbol{S}^d : \boldsymbol{S}^d\; d\Omega$$

where $\boldsymbol{S}^d$ is the deviatoric part of the strain rate tensor,

$$\epsilon_2 = \frac{\mu_v}{\rho_0} \frac{1}{\Omega} \int_\Omega (\nabla \cdot \boldsymbol{v})^2 \; d\Omega$$

where μ_v is the bulk viscosity and

$$\epsilon_3 = -\frac{1}{\rho_0 \, \Omega} \int_\Omega p \, \nabla \cdot \boldsymbol{v} \; d\Omega \; .$$

The second contribution is zero as the fluid is taken with $\mu_v = 0$. The third contribution is small as compressibility effects are small due to the small Mach number. The main contribution is thus the first one; given that the compressibility effects are small, this contribution can also be approximated using the enstrophy integral.

If possible, the temporal evolution of ϵ_1 and ϵ_3 shall also be reported, in addition to $-\frac{dE_k}{dt}$ and the enstrophy.

- participants are furthermore encouraged to provide numerical variants of kinetic energy dissipation rate (eg. including jump terms for DG methods) and compare those to the consistent values.

9 Reference data

The results will be compared to a reference incompressible flow solution. This solution has been obtained using a dealiased pseudo-spectral code (developed at Université catholique de Louvain, UCL) for which, spatially, neither numerical dissipation nor numerical dispersion errors occur; the time-integration is performed using a low-storage 3-steps Runge-Kutta scheme [2], with a dimensionless timestep of $1.0\,10^{-3}$. These results have been grid-converged on a 512^3 grid (a grid convergence study for a spectral discretization has also been done by van Rees et al. in[1]); this means that all Fourier modes up to the 256th harmonic with respect to the domain length have been captured exactly (apart from the time integration error of the Runge-Kutta scheme). The reference solutions are to be found in the following files:

- `spectral_Re1600_512.gdiag` provides the evolution of dimensionless values of E_k, $\epsilon = -\frac{dE_k}{dt}$ and $\mathcal{E}$.
- `wn_slice_x0_08000.out` provides the dimensionless vorticity norm on the plane $\frac{x}{L} = -\pi$. One can use the python script `read_and_plot_w.py` to extract the data and visualize the vorticity field or use another language following the format described in the script.

References

[1] W. M. van Rees W. M., A. Leonard, D. I. Pullin and P. Koumoutsakos, *A comparison of vortex and pseudo-spectral methods for the simulation of periodic vortical flows at high Reynolds numbers*, J. Comput. Phys., 230(2011), 2794-2805.

[2] J. H. Williamson, *Low-storage Runge-Kutta schemes*, J. Comput. Phys. 35(1980)