Radiation transport modeling using extended quadrature method of moments

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\textbf{Abstract}

The radiative transfer equation describes the propagation of radiation through a material medium. While it provides a highly accurate description of the radiation field, the large phase space on which the equation is defined makes it numerically challenging. As a consequence, significant effort has gone into the development of accurate approximation methods. Recently, an extended quadrature method of moments (EQMOM) has been developed to solve univariate population balance equations, which also have a large phase space and thus face similar computational challenges. The distinct advantage of the EQMOM approach over other moment methods is that it generates moment equations that are consistent with a positive phase space density and has a moment inversion algorithm that is fast and efficient. The goal of the current paper is to present the EQMOM method in the context of radiation transport, to discuss advantages and disadvantages, and to demonstrate its performance on a set of standard one-dimensional benchmark problems that encompass optically thin, thick, and transition regimes. Special attention is given in the implementation to the issue of realizability—that is, consistency with a positive phase space density. Numerical results in one dimension are promising and lay the foundation for extending the same framework to multiple dimensions.

\textbf{1. Introduction}

Radiation transport plays an important role in a wide range of physical applications. A very general application is to solar radiation [59,68], the most important source of energy for the earth. Another common application is in weather or climate models [13,76], where the radiative forcing is calculated for greenhouse gases, aerosols or clouds. Radiation transport modeling for supernovae [22,69,72] has been a topic of great interest for astrophysicists. With future interests in space travel, it finds another application in planetary reentry problems [1,67]. In biomedical sciences, the development of photon radiotherapy techniques [30,50] and related equipment rely heavily on radiation transport modeling. For the development of bio-renewable fuels, the design of photobioreactors [14,80] needs accurate radiation transport models for light absorption and scattering by microorganisms such as algae. In the environmental context, radiation transport modeling is important for the efficient design of furnaces [32,82] to minimize energy loss and pollution emissions.

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The current work focuses specifically on thermal radiative transfer [4,11,12,60,65] which involves solving two coupled equations. One is a complex integro-differential kinetic transport equation for the specific intensity of radiation and the other is an evolution equation describing the change in internal energy of the material through which the radiation is propagating. The transport equation is particularly challenging to solve because of the high-dimensionality of the phase space on which it is defined. A variety of different models have been proposed to numerically solve the transport equation, but there are essentially four major approaches: diffusion approximations, Monte Carlo, discrete-ordinates and moment methods.

In optically thick systems, the high rate of particle emission and absorption causes the radiation field to come into equilibrium with the material. The total energy of the coupled system is then characterized by the material temperature, and over long time scales, it satisfies a nonlinear diffusion equation up to an error that is proportional to the ratio of the photon mean free path to the macroscopic length scale. In optically thick systems, this ratio is small and the diffusion approximation [60,62,65] provides an accurate representation of the radiation field. However, in optically thin regimes, this ratio is large, and the diffusion approximation is no longer valid. Thus modifications have been introduced to extend the validity of the diffusion approximation into the transition regime, where emission and absorption are frequent enough to give structure to the radiation field, but not frequent enough to validate the diffusion limit. These include non-equilibrium diffusion [38,61] and flux-limited diffusion [34,46,47,75]. The latter is derived by enforcing the physical property that radiation cannot propagate faster than the speed of light, which has the effect of decreasing the diffusion coefficient near steep spatial gradients. Even with these extensions, the validity of the diffusion approximation does not extend much beyond optically thick regimes.

Monte Carlo methods [23,31,51,73] are the community standard-bearer with respect to accuracy, but with that accuracy comes a significant increase in both computational and memory requirements. Individual photons are simulated and then a statistical average is taken to determine the intensity. Due to the finite number of photons, statistical noise is an important issue in determining the accuracy of the computed moments.

The discrete-ordinates, or $S_n$ method [9,39,43,44,52,65,70] is the most popular deterministic method for numerically solving the transport equation. This method solves the transport equation along a discrete set of angular directions that are taken from a given quadrature set [10]. The quadrature set is then used to compute angular integrals. The main drawback of the method is the existence of numerical artifacts known as ray effects [43,45,49] that arise because the particles move only along directions in the quadrature set. These effects are most pronounced in optically thin materials with localized sources or sharp material discontinuities.

Moment models track the evolution of a finite number of weighted angular averages, or moments, of the specific intensity. They require a closure to make up for the information contained in the moments that are thrown away, and are typically derived by approximating the specific intensity as a function of the moments that remain in the model. In radiative transport, the most common moment closure is derived using a spherical harmonic expansion of the radiation intensity [16,34,49,52,65]. The result is a system of linear hyperbolic equations (often referred to as the $P_N$ equations) for the expansion coefficients. The spherical harmonic expansion does not preserve positivity of the intensity. As a consequence, the $P_N$ equations generate unphysical, negative radiation energy densities. However, they do preserve the rotational symmetry of the transport operator. It is this lack of symmetry which causes ray effects in the discrete-ordinates model.

Defects in the $P_N$ equations have led researchers to look for better closure models. Filtered $P_N$ closures [56,57] are less oscillatory than standard $P_N$ closures, but they do not completely remove the negative radiation energies. Positive $P_N$ closures [28] and entropy-based closures [29,20,21,26,5,7] do eliminate unphysical negative radiation energy densities. However, both of these closures typically require the numerical solution of a convex optimization problem at every point in a space-time mesh, and neither of them are well-defined for moments that are uniquely generated by atomic measures on the angular space, i.e. for moments that are uniquely generated by a specific intensity represented using delta functions in the angular space. In spite of these challenges, the search for practical closures which reproduce basic features of the kinetic equations continues. This is because the framework of moment methods makes them very flexible. Indeed, they can be used as stand-alone models, as members of an adaptive hierarchy, as preconditioners for higher resolution models, and as components of a hybrid description.

Recently, an extended quadrature method of moments (EQMOM) [83] was developed to solve univariate population balance equations in the context of particle transport in dispersed-phase flows. EQMOM has several desirable properties that make it a suitable candidate for radiation transport modeling. It is based on the quadrature method of moments (Q MOM) which, given a set of moments, represents the underlying distribution by a positive linear combination of delta functions. Thus, while similar to discrete ordinates, the location of the delta functions with QMOM is not predetermined. EQMOM uses smooth kernel density functions as basis functions, but it reduces to QMOM for those moments which can only be generated by atomic measures. Thus it can accurately reconstruct beams in any direction, but is otherwise expected to mitigate ray effects in multi-dimensional problems. Moreover, QMOM and EQMOM use a robust moment-inversion algorithm that guarantees the non-negativity of the specific intensity, and is much less expensive than solving a numerical optimization problem. Finally, like most moment methods, EQMOM preserves the hyperbolic nature of radiation transport.

The aim of the current paper is to lay a foundation for solving radiation transport problems using EQMOM. To this end, the paper applies the EQMOM framework to the equations of radiative transfer, and then tests it on a series of one-dimensional benchmark test problems. The current paper only discusses grey (frequency-independent) radiation transport. Non-grey cases will be considered in future work. The remainder of this paper is organized as follows. In Section 2, the equations of radiative transfer are discussed. Section 3 deals with quadrature-based closures. Thereafter, in Section 4, the solution
2. Equations of radiative transfer

The equation for time-dependent grey radiative transfer \([6,54]\) in a one-dimensional planar geometry medium with space and temperature dependent opacities is given by

\[
\frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial z} + \sigma_t I = \frac{1}{2} a c \sigma_a T^4 + \frac{1}{2} \sigma_s M_0 + \frac{1}{2} S, \tag{1}
\]

where \(I = I(z, \mu, t)\) is the specific intensity of radiation; \(\mu = \cos \theta \in [-1, 1]\) is the internal coordinate associated with the angle \(\theta \in [0, \pi]\) between a photon's direction of flight and the z-axis; \(\sigma_a = \sigma_a(z, T)\), \(\sigma_s = \sigma_s(z, T)\) and \(\sigma_t = \sigma_t(z, T)\) are the absorption, scattering and total opacities respectively; \(T = T(z, t)\) is the material temperature; and \(S = S(z)\) is an isotropic external source of radiation. The total opacity is the sum of the absorption and the scattering opacities \((\sigma_t = \sigma_a + \sigma_s)\). In (1), \(M_0 = M_0(z, t)\) is the zeroth-order moment of the specific intensity. The \(p\)th-order (monomial) moment of the specific intensity is defined as

\[
M_p = \int_{-1}^{1} \mu^p I(\mu) d\mu. \tag{2}
\]

For simplicity, in (2) and the following discussion, the explicit dependence of the specific intensity and the moments on spatial coordinate and time has been suppressed (i.e. \(M_p = M_p(z, t)\) and \(I(\mu) = I(z, \mu, t)\)). The evolution equation for the internal energy density \(e [6,54]\) of the material is given by

\[
\frac{\partial e}{\partial t} = \sigma_a M_0 - a c \sigma_a T^4. \tag{3}
\]

If the material is stationary, as will be assumed in this work, then \(e = e(T)\) and \(\partial e / \partial t = C_v \partial T / \partial t\) where \(C_v = C_v(T)\) is the heat capacity at constant volume. Applying the definition of moments in (2) to (1), the moment transport equation for the \(p\)th-order moment can be written as

\[
\frac{1}{c} \frac{\partial M_p}{\partial t} + \frac{\partial M_{p+1}}{\partial z} + \sigma_t M_p = \left\{ \frac{1}{2} a c \sigma_a T^4 + \frac{1}{2} \sigma_s M_0 + \frac{1}{2} S \right\} \left[ \frac{1 - (-1)^{p+1}}{p+1} \right]. \tag{4}
\]

Putting \(p = 0\) gives the transport equation for the zeroth-order moment:

\[
\frac{1}{c} \frac{\partial M_0}{\partial t} + \frac{\partial M_1}{\partial z} = a c \sigma_a T^4 - \sigma_a M_0 + S. \tag{5}
\]

\(M_0/c\) and \(M_1\) are often referred to as the radiation energy density and the radiation flux respectively. Note that in the absence of any external source of radiation \((S = 0)\), the total energy \(e + M_0/c\) is conserved:

\[
\frac{\partial e}{\partial t} + \frac{1}{c} \frac{\partial M_0}{\partial t} + \frac{\partial M_1}{\partial z} = 0. \tag{6}
\]

The discretized equations should satisfy a discrete version of this property.

The moment transport equations are not closed. The convection term in the evolution equation for each moment involves a higher-order moment. Thus to solve a finite set of moment transport equations, a closure model is needed. In the current work, EQMOM is used for closure. The details of EQMOM are discussed in the next section.

3. Quadrature-based closures

This section discusses quadrature-based closures. The current work uses EQMOM for closure, but since the moment-inversion algorithm for EQMOM is based on quadrature method of moments (QMOM), QMOM is presented first. The presentation here is brief and meant chiefly to make the paper self-contained. More details on QMOM can be found in [18,24,25,58], while a complete description of EQMOM can be found in [83].

3.1. Quadrature method of moments

For QMOM, the specific intensity is written as a weighted sum of delta functions:

\[
I(\mu) = \sum_{z=1}^{N} w_z \delta(\mu - \mu_z), \tag{7}
\]

where \(w_z\) and \(\mu_z\) are the weights and abscissas, respectively. The form of QMOM in (7) is referred to as \(N\)-node QMOM as it uses \(N\)-delta functions to represent the specific intensity. Using (7), the moments in (2) can be written as
\[ M_p = \sum_{s=1}^{N} W_s (\mu_s)^p. \]

The first 2N moments \( \{M_0, M_1, \ldots, M_{2N-1}\} \) are tracked and the N weights and N abscissas are computed by solving the set of 2N non-linear equations, obtained from (8), using the Wheeler algorithm [58,81,83]. Other robust algorithms for computing the weights and abscissas are discussed in [33,77]. Once the moments are inverted to recover the weights and abscissas, the desired unknown moments can be approximated using (7). It is worth mentioning that the inversion will succeed only if the moment set \( \{M_0, M_1, \ldots, M_{2N-1}\} \) is realizable—that is, only if each component satisfies (8) for some non-negative distribution on \([-1, 1]\). If the moment set is non-realizable, the inversion will fail, giving negative weights and/or abscissas that lie outside the support interval \([-1, 1]\). In order to ensure the success of the inversion algorithm, the realizability of the moment set is guaranteed using realizable discretization schemes discussed in Section 4. These realizable discretization schemes guarantee that the moment set corresponds to a non-negative specific intensity at all times.

Note that the form (7) is similar to the one used by the discrete-ordinates model. However, the two models are different in the way in which the abscissas (angular directions) are chosen. In the discrete-ordinates model, the abscissas belong to a pre-specified quadrature set. For the one-dimensional setting, a Gauss-Legendre set is most often used because it is optimal in terms of polynomial integration. In particular, an N-point Gauss-Legendre quadrature will (in exact arithmetic) integrate polynomials of degree up to 2N – 1 exactly, i.e., it can accommodate 2N moments with respect to the weight function one. However, only N moments can be reconstructed with respect to a general weight function. QMOM, on the other hand, treats both the abscissas and weights as unknowns, and both are computed as a part of the inversion algorithm. Thus a representation given by N nodes can be used to specify 2N moments with respect to an unknown weight function—in this case, the underlying specific intensity.

From the theory of moments [15,19,71], the specific intensity corresponding to a moment set on the boundary of moment space (the space of moments which are realizable from a non-negative distribution) can be written as a weighted sum of delta functions. As QMOM uses delta functions as its basis functions, it can successfully reconstruct the specific intensity when the moment set lies on the boundary of the moment space. This property does not hold for some other popular moment methods. On the other hand, if the moment set \( \{M_0, M_1, \ldots, M_{2N-1}\} \) lies in the interior of the moment space, the value of \( M_{2N} \) computed using N-node QMOM is the smallest value possible, meaning that the specific intensity reconstructed using N-node QMOM has moments of order higher than 2N – 1 that lie on the boundary of moment space.

Despite a fast and robust moment-inversion algorithm, QMOM has some drawbacks. The 2N eigenvalues for the N-node closure are not unique, and thus the moment system is weakly hyperbolic. For example, in the 2-node case, the set of eigenvalues is \( \{\mu_1, \mu_1, \mu_2, \mu_2\} \). This weakly hyperbolic nature of the equations leads to delta shocks in the radiation energy density which are unphysical. Moreover, just like the discrete-ordinates model, QMOM allows radiation transport only along a finite number of angular directions. Another problem with QMOM is that point-wise values of the specific intensity are not always guaranteed using realizable discretization schemes discussed in Section 4. These realizable discretization schemes guarantee that the moment set corresponds to a non-negative specific intensity at all times.

3.2. Extended quadrature method of moments

EQMOM was developed to utilize the fast and robust inversion algorithm of QMOM while, at the same time, replacing its less appealing features with the better ones borrowed from kernel density element method (KDEM) [2]. Unlike QMOM, which represents the specific intensity as a weighted sum of delta functions, EQMOM uses a weighted sum of known non-negative kernel density functions, which is similar to the representation used for KDEM. At the same time, EQMOM avoids solving the multivariate constrained least-squares problem of KDEM and instead requires only a one-dimensional root-finding method based on QMOM inversion algorithm. Moreover, EQMOM reduces to QMOM in a well-conditioned manner, making it possible to easily reconstruct the specific intensity all the way to the boundary of the moment space. In what follows, the form of the specific intensity used for EQMOM and the corresponding inversion algorithm are described.

3.2.1. Transformed moments

As is discussed later, EQMOM in the current work employs beta distribution functions, which are usually defined on the interval \([0, 1]\). Taking this into consideration, a new angular variable \( \tilde{\mu} = (\mu + 1)/2 \in [0, 1] \) and a new specific intensity \( \tilde{I}(\tilde{\mu}) = 2I(2\tilde{\mu} - 1) \) are defined. Transformed moments \( \tilde{M}_p = \int_0^1 \tilde{I}^{p+1}(\tilde{\mu}) d\tilde{\mu} \) of \( \tilde{I} \) can be computed from the original moments using the formula

\[ \tilde{M}_p = \frac{1}{2^p} \sum_{l=0}^{p} \binom{p}{l} M_l. \]

In the following discussion, the transformed moments \( \tilde{M}_p \) are used instead of the actual moments \( M_p \).

EQMOM uses two successive quadrature approximations to reconstruct the specific intensity from the tracked moments. These successive approximations are referred to as primary and secondary quadratures. In the discussion below, N nodes are used for primary quadrature and M nodes are used for secondary quadrature per primary node. The primary quadrature is discussed first.
3.2.2. Primary quadrature

For the N-node primary quadrature, the transformed specific intensity is represented as a weighted sum of N smooth kernel density functions:

\[
\overline{I}(\overline{\mu}) = \sum_{x=1}^{N} w_x G_{p}(\overline{\mu}, \overline{\mu}_x)
\]  

(10)

where \(w_x\) and \(\overline{\mu}_x, \alpha \in \{1, 2, \ldots, N\}\), are the primary quadrature weights and abscissas, respectively, and \(\zeta\) is a measure of the spread of the individual smooth kernel density functions. Note that the same \(\zeta\) is used for each of the N kernel density functions. In this work, each smooth kernel density function is chosen to be a beta distribution function [79]:

\[
\delta_{p}(\overline{\mu}, \overline{\mu}_x) = \frac{1}{B(\xi_x, \eta_x)} \overline{\mu}^{\zeta-1}(1 - \overline{\mu})^{\eta_x-1},
\]  

(11)

where \(B(\xi_x, \eta_x)\) is the beta function and the two parameters, \(\xi_x = \overline{\mu}_x / \zeta\) and \(\eta_x = (1 - \overline{\mu}_x) / \zeta\), depend on the value of \(\zeta\). This form of EQMOM is usually called beta-EQMOM, but in the current paper it will simply be referred to as EQMOM. Note that with appropriate choice of parameters (\(\xi_x = 1\) and \(\eta_x = 1\)), a uniform distribution in the \(\mu\)-space (or \(\mu\)-space) can be recovered. In order to reconstruct \(I, 2N + 1\) parameters (\(\{w_x, \overline{\mu}_x\}\) for \(\alpha \in \{1, 2, \ldots, N\}\) and \(\zeta\)) are needed. Note that this is one more than the number of unknown parameters in the N-node QMOM. To find these \(2N + 1\) unknowns, the first \(2N + 1\) transformed moments (\(\{\overline{M}_0, \overline{M}_1, \ldots, \overline{M}_{2N}\}\)) are used which are obtained by tracking the first \(2N + 1\) actual moments (\(\{M_0, M_1, \ldots, M_{2N}\}\)).

Using (11), the transformed moments can be written as

\[
\overline{M}_0 = \sum_{x=1}^{N} w_x, \quad \text{and} \quad \overline{M}_p = \sum_{x=1}^{N} w_x G_{p}(\mu_x, \zeta) \quad \text{for} \quad p \geq 1.
\]  

(12)

Define the star moments as \(\overline{M}_{\ast} = \sum_{x=1}^{N} w_x (\overline{\mu}_x)^p\). This definition is crucial as it facilitates the use of QMOM inversion algorithm. Note that the definition of star moments is similar to the form of moments used in QMOM. Substituting (13) in (12) and using the definition of star moments, the transformed moments can be related to the star moments as \(\overline{M}_0 = \overline{M}_0^\ast\) and \(\overline{M}_p = \gamma_p \overline{M}_p^\ast + \gamma_{p-1} \overline{M}_{p-1}^\ast + \cdots + \gamma_1 \overline{M}_1^\ast\) for \(p \geq 1\) [83], where the non-negative coefficients \(\gamma_p\) depend only on \(\zeta\). For the first five transformed moments and the corresponding star moments can be written as

\[
\begin{align*}
\overline{M}_0 &= \overline{M}_0^\ast, \\
\overline{M}_1 &= \overline{M}_1^\ast, \\
\overline{M}_2 &= \frac{1}{(1 + \zeta)} (\overline{M}_2^\ast + \zeta \overline{M}_1^\ast), \\
\overline{M}_3 &= \frac{1}{(1 + 2\zeta)(1 + \zeta)} (\overline{M}_3^\ast + 3\zeta \overline{M}_2^\ast + 2\zeta^2 \overline{M}_1^\ast), \\
\overline{M}_4 &= \frac{1}{(1 + 3\zeta)(1 + 2\zeta)(1 + \zeta)} (\overline{M}_4^\ast + 6\zeta \overline{M}_3^\ast + 11\zeta^2 \overline{M}_2^\ast + 6\zeta^3 \overline{M}_1^\ast).
\end{align*}
\]

(14)

It is worth observing that this is a lower triangular system and hence the star moments can be easily computed from the transformed moments if a value of \(\zeta\) is specified.

The algorithm for computing the \(2N + 1\) unknown parameters in the primary quadrature utilizes the moment-inversion algorithm for QMOM. Instead of treating all the \(2N + 1\) parameters as unknowns, only \(\zeta\) is considered to be the unknown and the N weights and N abscissas are computed as a function of \(\zeta\) using the QMOM moment-inversion algorithm. This leads to a fast inversion algorithm for EQMOM which is based on one-dimensional root-finding to compute \(\zeta_{\text{optimal}}\). A target function, \(J_{2N}(\zeta) = \overline{M}_{2N} - (\gamma_{2N} \overline{M}_{2N}^\ast + \gamma_{2N-1} \overline{M}_{2N-1}^\ast + \cdots + \gamma_1 \overline{M}_1^\ast)\), is defined and \(\zeta_{\text{optimal}}\), that corresponds to \(J_{2N}(\zeta_{\text{optimal}}) = 0\), is searched for. The steps involved in the algorithm are as follows:

1. Set \(\zeta = 0\).
2. Using \(\{\overline{M}_0, \overline{M}_1, \ldots, \overline{M}_{2N-1}\}\) and \(\zeta\), compute \(\{\overline{M}_0^\ast, \overline{M}_1^\ast, \ldots, \overline{M}_{2N-1}^\ast\}\).
3. Using \(\{\overline{M}_0^\ast, \overline{M}_1^\ast, \ldots, \overline{M}_{2N-1}^\ast\}\), compute \(\{\overline{w}_x, \overline{\mu}_x\} \forall \alpha \in \{1, 2, \ldots, N\}\). This is done using the moment-inversion algorithm for QMOM.
4. Using \(\{\overline{w}_x, \overline{\mu}_x\} \forall \alpha \in \{1, 2, \ldots, N\}\), compute \(\overline{M}_{2N} = \sum_{x=1}^{N} \overline{w}_x (\overline{\mu}_x)^{2N}\).
5. Using \(\overline{M}_{2N}\) and \(\{\overline{M}_0^\ast, \overline{M}_1^\ast, \ldots, \overline{M}_{2N}^\ast\}\), compute \(J_{2N}(\zeta)\).
6. If $J_{2N}(\zeta) = 0$, the corresponding $\zeta$ is $\zeta_{\text{optimal}}$, which along with the weights and abscissas obtained in Step 3 provides the primary quadrature information. Else, make another guess for $\zeta$ using the approach described below and return to Step 2.

Note that the above algorithm guarantees that the first $2N$ transformed moments are exactly reproduced for each value of $\zeta$. And for $\zeta_{\text{optimal}}$, all the $2N + 1$ transformed moments are reproduced exactly. In Step 6, a combination of secant and bisection methods is used to improve the guessed value of $\zeta$. Note that the algorithm starts with $\zeta = 0$. According to the theory of moments [15, 19, 71], $J_{2N}(0) \geq 0$. If $J_{2N}(0) = 0$, the moment set lies on the boundary of the moment space. For this case the above algorithm needs only one iteration and the EQMOM solution reduces to the QMOM solution. However, if $J_{2N}(0) > 0$, the secant method is used to make another guess for $\zeta$. The secant method needs two values of $\zeta$ as input. At the start of the iteration, the first value is $\zeta = 0$ and a second very small value is specified. The secant method is used until a value of $\zeta$ with $J_{2N}(\zeta) < 0$ is obtained. When this happens, $\zeta_{\text{optimal}}$ is bracketed and the bisection method is used thereon. Note that the bisection method guarantees the determination of $\zeta_{\text{optimal}}$ while using the secant method only does not provide this guarantee.

3.2.3. Realizability of star moments

In the algorithm described above, Step 3 is based on the moment-inversion algorithm for QMOM and will succeed only if the moments $\{M_0, M_1, \ldots, M_{2N-1}\}$ form a realizable set. The star moment set $\{M_0^*, M_1^*, \ldots, M_{2N-1}^*\}$ is computed from the transformed moment set $\{M_0, M_1, \ldots, M_{2N-1}\}$ using the guessed value of $\zeta$. The realizability of the actual moment set and hence the transformed moment set is guaranteed by using the realizable discretization schemes discussed in Section 4. However, even a realizable transformed moment set can lead to a non-realizable star moment set for some values of $\zeta$, thereby causing the failure of the QMOM inversion algorithm. In order to check the realizability of the star moment set to know whether the QMOM inversion will succeed, every time after Step 2 the realizability of the star moment set is checked using the Hankel determinants [15, 19, 71, 83]:

$$
H_{2k} = \begin{bmatrix}
\tilde{M}_0 & \cdots & \tilde{M}_k \\
\vdots & \ddots & \vdots \\
\tilde{M}_k & \cdots & \tilde{M}_{2k}
\end{bmatrix}, \quad 
H_{2k+1} = \begin{bmatrix}
\tilde{M}_1 & \cdots & \tilde{M}_{k+1} \\
\vdots & \ddots & \vdots \\
\tilde{M}_{k+1} & \cdots & \tilde{M}_{2k+1}
\end{bmatrix}
$$
$$
\Pi_{2k+1} = \begin{bmatrix}
\tilde{M}_0 - \tilde{M}_1 & \cdots & \tilde{M}_k - \tilde{M}_{k+1} \\
\vdots & \ddots & \vdots \\
\tilde{M}_k - \tilde{M}_{k+1} & \cdots & \tilde{M}_{2k} - \tilde{M}_{2k+1}
\end{bmatrix}, \quad 
\Pi_{2k} = \begin{bmatrix}
\tilde{M}_1 - \tilde{M}_2 & \cdots & \tilde{M}_k - \tilde{M}_{k+1} \\
\vdots & \ddots & \vdots \\
\tilde{M}_k - \tilde{M}_{k+1} & \cdots & \tilde{M}_{2k} - \tilde{M}_{2k+1}
\end{bmatrix}
$$

All the Hankel determinants for $k = 0, \ldots, N - 1$ are checked. If any of the Hankel determinants is negative, the star moment set is non-realizable and the QMOM algorithm will not succeed. For such a case, $\zeta_{\text{optimal}}$ cannot be obtained using the EQMOM moment-inversion algorithm. Fig. 1 shows two cases, one of which succeeds in finding $\zeta_{\text{optimal}}$ using the EQMOM inversion algorithm while the other fails. Note that multiple $\zeta_{\text{optimal}}$ may be possible. The current implementation of EQMOM chooses the smallest $\zeta_{\text{optimal}}$. In case any of the Hankel determinants is negative and the QMOM inversion algorithm fails, out of all the guessed values of $\zeta$, the one which gives the smallest $J_{2N}(\zeta)$ is chosen. Note that the first 2N moments ($\{M_0, M_1, \ldots, M_{2N-1}\}$) are still exactly reproduced. Only the highest order moment ($M_{2N}$) is not reproduced exactly. However, it is worth mentioning that this realizability failure occurs very rarely.

3.2.4. Secondary quadrature

Substituting (10) into the definition of transformed moments yields

$$
\tilde{M}_p = \sum_{z=1}^{N} \bar{w}_z \int_0^1 \tilde{\mu}_p \delta_z(\tilde{\mu}, \tilde{\mu}_z)d\tilde{\mu}.
$$

(16)

Note that for the $N$-node primary quadrature $p \in \{0, 1, \ldots, 2N\}$. As the kernel density functions, $\delta_z(\tilde{\mu}, \tilde{\mu}_z)$, are beta distributions, using Gauss-Jacobi quadrature the following equality holds for all $p \in \{0, 1, \ldots, 2N\}$ as long as $M \geq N + 1$:

$$
\int_0^1 \tilde{\mu}_p \delta_z(\tilde{\mu}, \tilde{\mu}_z)d\tilde{\mu} = \sum_{\beta=1}^{M} \bar{p}_{2\beta} \delta(\tilde{\mu} - \tilde{\mu}_{2\beta}),
$$

(17)

where $\bar{p}_{2\beta}$ and $\tilde{\mu}_{2\beta}$ are the weights and abscissas for Gauss–Jacobi quadrature. Note that this is equivalent to writing $\delta_z(\tilde{\mu}, \tilde{\mu}_z) = \sum_{\beta=1}^{M} \bar{p}_{2\beta} \delta(\mu - \tilde{\mu}_{2\beta})$ as it reproduces all the moments exactly. Substituting this in (10) gives the dual-quadrature representation:

$$
\tilde{I}(\tilde{\mu}) = \sum_{z=1}^{N} \sum_{\beta=1}^{M} \bar{w}_{2\beta} \delta(\tilde{\mu} - \tilde{\mu}_{2\beta})
$$

(18)
and occurs in the flux term \( \frac{1}{C_0} \) and \( \frac{1}{C_1} \) that, for the radiation transport problem, the highest order moment that needs closure is some polynomial moments, they can be evaluated to arbitrary accuracy by increasing the value of \( M \). Increasing the value of \( M \) should be expected if \( M \) is increased beyond \( N + 1 \). However, there is another reason which dictates the use of a larger \( M \). As mentioned later in Section 4, the realizable discretization schemes for convection terms use a high-order reconstruction for weights but only first-order (piecewise constant) reconstruction for abscissas. These schemes are quasi-high-order accurate and can attain close to formal high-order of accuracy only if the abscissas are smoothly varying. Increasing the value of \( M \) increases the number of abscissas and decreases the spatial variation in each abscissas thereby making them smoothly varying and increasing the overall accuracy of the scheme.

The dual-quadrature representation of the actual specific intensity corresponding to (18) can be written as

\[
I(\mu) = \sum_{a=1}^{N} \sum_{b=1}^{M} w_{ab} \delta(\mu - \mu_{ab})
\]

where \( w_{ab} = \bar{w}_{ab} \bar{\mu}_{ab} \) and \( \mu_{ab} = 2 \bar{\mu}_{ab} - 1 \). This form is used to evaluate the unclosed terms in the moment transport equations. The whole algorithm discussed above can be summarized as

\[
\begin{align*}
&M_0, M_1, \ldots, M_{2N} \\
&\mu\text{-space to } \mu\text{-space} \\
&[M_0, M_1, \ldots, M_{2N}] \\
&\quad \rightarrow \\
&[\bar{M}_0, \bar{M}_1, \ldots, \bar{M}_{2N}]
\end{align*}
\]

\[
\begin{align*}
&M_0, M_1, \ldots, M_{2N} \\
\text{Primary quadrature} \\
&M_0, M_1, \ldots, M_{2N} \\
\text{Secondary quadrature} \\
&M_0, M_1, \ldots, M_{2N} \\
&\mu\text{-space to } \mu\text{-space} \\
&[\bar{w}_{ab}, \bar{\mu}_{ab}] \\
&[w_{ab}, \bar{\mu}_{ab}]
\end{align*}
\]

Before moving to the next section, here is a brief summary of the properties that make EQMOM a viable option for radiation transport modeling. EQMOM uses smooth kernel density functions as basis functions, hence it allows radiation transport in all directions. This will help mitigate the ray effects in multi-dimensional problems. EQMOM has a robust inversion algorithm that avoids the computational cost of solving an expensive optimization problem and instead utilizes a novel combination of QMOM inversion algorithm and a one-dimensional root-finding method. Furthermore, for the moments lying on the boundary of the moment space, EQMOM reduces to QMOM in a well-conditioned manner. When EQMOM is used in conjunction with the realizable discretization schemes (described in Section 4), the non-negativity of the specific intensity is guaranteed, thereby avoiding any negative radiation energy density problem. The system of evolution equations for EQMOM have distinct eigenvalues that preserve the hyperbolic nature of radiation transport. Moreover as shown in Section 5, EQMOM can achieve the maximum radiation transport speed of \( c \) even with a single primary node.

4. Solution algorithm

The aim of the current paper is to demonstrate the effectiveness of EQMOM for radiation transport modeling. Development of efficient spatial and temporal discretization schemes for EQMOM applied to radiation transport modeling will be
considered as a part of future work. In the current work, the equations are discretized using the finite-volume approach [48]. A semi-implicit time-integration scheme, adapted from [54] is used. The convection and the external source terms are treated explicitly while the material coupling terms are linearized and treated implicitly. The conservation of the net energy is satisfied in discrete form. In addition, the realizability of the moment set is guaranteed at each step using appropriate realizable schemes and realizability criteria. This is essential for the success of the moment-inversion algorithm. The details of the treatment of different terms are given below. First the convection term is treated, followed by the external source term and eventually the effect of the material coupling terms is included:

\[
M_{p,l}^{(1)} - M_{p,l}^{(2)} + M_{p,l}^{(3)} \rightarrow M_{p,l}^{(4)}.
\]

4.1. Convection term

For the convection term, the first-order Euler explicit time-integration is used. The discretized equation can be written as:

\[
M_{p,i}^{(1)} = M_{p,i}^{n} - \frac{\Delta t}{\Delta z} \left[ G_{p,i+1/2}^{n} - G_{p,i-1/2}^{n} \right],
\]

where \( M_{p,i}^{n} \) is the pth-order cell-averaged moment in the ith cell at time level n, \( \Delta t \) is the time step size and \( \Delta z \) is the cell size. The quantities \( G_{p,i+1/2}^{n} \) and \( G_{p,i-1/2}^{n} \) are the numerical fluxes at the faces \( i + 1/2 \) and \( i - 1/2 \) respectively. The numerical flux is evaluated using a kinetic approach [17, 63, 78]:

\[
G_{p,i+1/2}^{n} = \int_{-1}^{0} \mu P_{i+1/2}^{n} d\mu + \int_{0}^{1} \mu P_{i+1/2}^{n} d\mu,
\]

where \( P_{i+1/2}^{n} \) and \( P_{i-1/2}^{n} \) are the reconstructed specific intensities at the left and right sides of the face \( i + 1/2 \) respectively. The specific intensity in the ith cell can be written as

\[
l_{i} = \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \delta(\mu - \mu_{\rho,\beta}^{n}),
\]

where \( w_{\rho,\beta}^{n} \) and \( \mu_{\rho,\beta}^{n} \) are the secondary weights and abcissas in the ith cell respectively. The reconstructed specific intensities at the left and right sides of the face \( i + 1/2 \) can be written as

\[
l_{i+1/2,l}^{n} = \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \delta(\mu - \mu_{\rho,\beta}^{n}) \quad \text{and}
\]

\[
l_{i+1/2,r}^{n} = \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \delta(\mu - \mu_{\rho,\beta}^{n})
\]

respectively. Using (25), the numerical flux at the face \( i + 1/2 \) is written as:

\[
G_{p,i+1/2}^{n} = \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \min(\mu_{\rho,\beta}^{n}, \mu_{\rho,\beta}^{n}) \min(\mu_{\rho,\beta}^{n}, \mu_{\rho,\beta}^{n}) + \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \max(\mu_{\rho,\beta}^{n}, \mu_{\rho,\beta}^{n}, 0) \max(\mu_{\rho,\beta}^{n}, \mu_{\rho,\beta}^{n}, 0). \]

The weights and abcissas at the left and right sides of the faces are reconstructed using the cell values. To guarantee the realizability of the moment set, only piecewise constant reconstruction of abcissas is allowed [78]:

\[
\mu_{\rho,\beta}^{n} = \mu_{\rho,\beta}^{n} \quad \text{and} \quad \mu_{\rho,\beta}^{n} = \mu_{\rho,\beta}^{n}. \]

As the abcissas vary from one cell to another, use of higher order reconstruction for abcissas does not guarantee the realizability of the transported moment set [78]. For the weights however, any high-order reconstruction can be used. In the current paper, piecewise linear reconstruction is used with minmod limiting [37] to avoid any spurious oscillations. This leads to the quasi-2nd-order scheme in [78]. The quasi-2nd-order scheme can achieve close to formal 2nd-order accuracy only if the spatial variation of abcissas is smooth. The greater the number of abcissas, smoother is the spatial variation in each of the abcissas. This is achieved by increasing the number of secondary quadrature nodes. Note that all the moments updated using (22) correspond to a unique update of the specific intensity:

\[
l_{i}^{1} = \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} \left[ w_{\rho,\beta}^{n} - \frac{\Delta t}{\Delta z} \left( w_{\rho,\beta}^{n}, 0 \right) \min(\mu_{\rho,\beta}^{n}, 0) \delta(\mu - \mu_{\rho,\beta}^{n}) - \frac{\Delta t}{\Delta z} \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \max(\mu_{\rho,\beta}^{n}, 0) \delta(\mu - \mu_{\rho,\beta}^{n}) \right] \delta(\mu - \mu_{\rho,\beta}^{n}) \times \min(\mu_{\rho,\beta}^{n}, 0) \delta(\mu - \mu_{\rho,\beta}^{n}) + \frac{\Delta t}{\Delta z} \sum_{\beta=1}^{N} \sum_{\rho=1}^{M} w_{\rho,\beta}^{n} \max(\mu_{\rho,\beta}^{n}, 0) \delta(\mu - \mu_{\rho,\beta}^{n}).
\]

The non-negativity of the updated specific intensity can be guaranteed if the time step size satisfies the realizability criterion given by
\[
\Delta t \leq \frac{w_{n+1/2,j}^0 \Delta z}{c \left( w_{n+1/2,j}^0 \max(\mu_{n+1/2,j}^0, 0) - w_{n-1/2,j}^0 \min(\mu_{n-1/2,j}^0, 0) \right)}
\] (29)

\(\forall n \in \{1, 2, \ldots, N\} \) and \(\beta \in \{1, 2, \ldots, M\}\). More details about the realizability criterion can be found in [78]. In addition to the realizability criterion, the time step size should satisfy the CFL condition for stability:

\[
\Delta t \leq CFL \frac{\Delta z}{c \mu_{n+1/2,j}^0}
\] (30)

\(\forall n \in \{1, 2, \ldots, N\} \) and \(\beta \in \{1, 2, \ldots, M\}\). Note that (29) is exactly same as (30) if piecewise constant reconstruction is used for weights instead of piecewise linear reconstruction. In the current work, CFL = 0.3 is used for all the numerical simulations, unless otherwise stated.

4.2. External source term

In the current work, the external source is only a function of spatial coordinate. It is treated using the first-order Euler explicit time-integration scheme:

\[
M_p^{(2)} = M_p^{(1)} + \frac{c \Delta t}{2} S_i \left[ 1 - \frac{(1)^{p+1}}{p+1} \right].
\] (31)

The updated moment set corresponds to the following unique update of the specific intensity:

\[
i_i^{(2)} = i_i^{(1)} + \frac{c \Delta t}{2} S_i.
\] (32)

The non-negativity of the updated specific intensity is guaranteed as long as the external source is non-negative.

4.3. Material coupling terms

The algorithm for the implicit treatment of the material coupling terms is adapted from [54]. The moment transport equation in (4), without the convection and the external source terms, and the material energy equation in (3) are discretized using the first-order Euler implicit time-integration scheme:

\[
\frac{M_{p,i}^{n+1} - M_{p,i}^{n+1}}{c \Delta t} = -\sigma_{n,i}^a M_{p,i}^{n+1} + \left\{ \frac{1}{2} ac \sigma_{n,i}^a (T_i^{n+1})^4 + \frac{1}{2} \sigma_{n,i}^a (T_i^{n+1})^4 \right\} \left[ 1 - \frac{(1)^{p+1}}{p+1} \right].
\] (33)

\[
\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \sigma_{n,i}^a (T_i^{n+1})^4 - ac \sigma_{n,i}^a (T_i^{n+1})^4.
\] (34)

Note that the opacities are treated explicitly. For \(p = 0\), (33) becomes

\[
\frac{M_{0,i}^{n+1} - M_{0,i}^{n+1}}{c \Delta t} = -\sigma_{n,i}^a (T_i^{n+1})^4 + ac \sigma_{n,i}^a (T_i^{n+1})^4.
\] (35)

The nonlinearities in the \(T_i^n\) terms are treated using the equivalent of one Newton iteration. To perform this linearization, \((T_i^{n+1})^4\) and \(\varepsilon_i^{n+1}\) are expanded using Taylor series:

\[
(T_i^{p+1})^4 \approx (T_i^n)^4 + 4(T_i^n)^3(T_i^{p+1} - T_i^n),
\] (36)

\[
\varepsilon_i^{n+1} \approx \varepsilon_i^n + C_i^n (T_i^{p+1} - T_i^n).
\] (37)

Substituting (36) in (37) and then putting the result in (34) yields

\[
(T_i^{p+1})^4 \approx \int_{T_i^n}^{T_i^{p+1}} \left[ \frac{f_{n}(T_i^n)^4 + \sigma_{n,i}^a \beta_i^n \Delta M_{0,i}^{n+1}}{ac(T_i^n)^4} \right].
\] (38)

where \(\beta_i^n = 4ac(T_i^n)^3/C_i^n\) and \(f_{n} = 1/\Gamma_{n,i}(\beta_i^n \Delta t)\). Substituting (38) in (33) gives

\[
\frac{M_{p,i}^{n+1} - M_{p,i}^{n+1}}{c \Delta t} = -\sigma_{n,i}^a (T_i^{n+1})^4 + \left\{ \frac{1}{2} ac \sigma_{n,i}^a (T_i^n)^4 + \frac{1}{2} \sigma_{n,i}^a (T_i^n)^4 \right\} \left[ 1 - \frac{(1)^{p+1}}{p+1} \right].
\] (39)

Putting \(p = 0\) and performing some simplifications, following equation for the zeroth-order moment is obtained:

\[
\frac{M_{0,i}^{n+1} - M_{0,i}^{n+1}}{c \Delta t} = -f_{n} \sigma_{n,i}^a [M_{0,i}^{n+1} - ac(T_i^n)^4].
\] (40)
This equation is used for updating the zeroth-order moment. Rest of the moments are then updated using (39). Substituting (38) in (34) yields

$$
\frac{e^{n+1}_i - e^n_i}{\Delta t} = \int_{\Sigma} n \sigma a_i \left[ M^{n+1}_{0,i} - ac(T^n_i) \right].
$$

(41)

This is used to update the material internal energy density. Note that

$$
\frac{M^{n+1}_{0,i} - M^{(2)}_{0,i}}{c\Delta t} + \frac{e^{n+1}_i - e^n_i}{\Delta t} = 0.
$$

(42)

Hence, the discretization method conserves the net energy. Using (39), the unique update of the specific intensity can be written as

$$
I^{n+1}_i = \frac{1}{1 + c\Delta t \sigma i} \left[ I^{n+2}_i + \frac{1}{2} c\Delta t \sigma i \left( \frac{ac(T^n_i)^4 + \sigma e_i^r n M^{n+1}_{0,i}}{\Delta t M^{n+1}_{0,i}} \right) + \frac{1}{2} c\Delta t \sigma i M^{n+1}_i \right].
$$

(43)

All the terms on the right hand side are non-negative, thereby guaranteeing the non-negativity of the updated specific intensity and hence the realizability of the moment set at the time level $n + 1$.

5. Numerical results

In this section, results are presented for some standard benchmark problems encompassing optically thin, thick and transition regimes. For simplicity, the units of all the quantities are given in Table 1, and hereafter all the units will be dropped when specifying their values. The domain of the problems is defined by $z \in [z_{LB}, z_{RB}]$. Material coupling is only required for the last three problems: the diffusive and non-equilibrium Marshak wave problems and the Su–Olson problem. The two constants used in the problems are the speed of light ($c = 3 \times 10^{10}$ cm/s) and the radiation constant ($a = 1.372 \times 10^{14}$ erg/cm$^3$/keV$^2$). For all the problems, results are presented for EQMOM with different number of primary nodes. For the current implementation of EQMOM, a maximum of 5 primary nodes is used. Round-off errors become dominant when more than 5 primary nodes are used. Improvements will be made to future implementations to allow the use of more than 5 primary nodes. For each primary node, the number of secondary nodes have been chosen such that any further increase leaves the solution almost unchanged. Note that as mentioned in Section 3.2.4, although $M = N + 1$ is sufficient to reproduce all the moments, an increased value of $M$ helps improve the order of accuracy of the spatial discretization scheme. In all the discussions below, EQMOM-NxM refers to EQMOM with $N$ primary nodes and $M$ secondary nodes per primary node. It is worth reiterating that for EQMOM with $N$ primary nodes, the first $2N + 1$ moments are tracked. For all the problems, results are presented on successively refined grids to show the general trend of grid convergence of the EQMOM solutions.

5.1. Plane-source problem

The plane-source problem [5,27,29] is a torture test for a model to handle very strong spatial discontinuities. The scattering and absorption opacities are given by $\sigma_s = 1$ and $\sigma_a = 0$ respectively. There is no external source of radiation ($S = 0$). The specific intensity and the corresponding moments at $t = 0$ are given by

$$
I(z, \mu) = 0.5\delta(z) + 0.00005 \quad \text{and} \quad M_p(z) = (0.5\delta(z) + 0.00005) \left[ \frac{1 - (-1)^{p+1}}{p + 1} \right]
$$

(44)

respectively. Note that a small positive baseline value is used as in [29]. However, the method works with zero baseline value as well. The entropy-based closures use exponential basis functions and hence cannot use the zero baseline value. The left boundary is located at $z_{LB} = 0$. Three different cases are simulated with final times of $ct = 1, 2$ and 4 with the corresponding right boundaries located at $z_{RB} = 1.1, 2.1$ and 4.1 respectively. Reflective boundary condition is applied at the left boundary:

$$
I_{LB}(-\mu) = I_{LB}(-\mu) \quad \text{and} \quad M_{p,LB} = (-1)^{p}M_{p,LB}.
$$

(45)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
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</tr>
<tr>
<td>$\sigma_s$</td>
<td>1/cm</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>1/cm</td>
</tr>
<tr>
<td>$t$</td>
<td>s</td>
</tr>
<tr>
<td>$T$</td>
<td>keV</td>
</tr>
<tr>
<td>$e$</td>
<td>erg/cm$^3$</td>
</tr>
<tr>
<td>$M_p$</td>
<td>erg/cm$^2$/s</td>
</tr>
</tbody>
</table>

Table 1:

Quantities and units.
Fig. 2. Solutions for the plane-source problem at $ct = 1$. 
Fig. 3. Solutions for the plane-source problem at $ct = 2$. 
At the right boundary, Dirichlet boundary condition with a fixed baseline specific intensity value is applied:

\[ I_{RB} (t) = 0.00005 \quad \text{and} \quad M_{p, RB} = 0.00005 \left[ \frac{1 - (-1)^{p+1}}{p + 1} \right] . \tag{46} \]

Note that the boundary conditions are applied at the left side of the leftmost face and the right side of the rightmost face. For this problem, results are presented with the number of primary nodes varying from 1 through 5. Fig. 2 shows the results at \( ct = 1 \) with 220, 440 and 880 cells. The semi-analytic solution is obtained from [27]. The oscillations are wave effects which arise when approximating the radiation field as a finite number of moments. The speed of these waves is determined by the finite number of eigenvalues of the system. The spherical harmonics model shows similar oscillations, however the oscillations there are more pronounced [29]. Increasing the number of primary nodes increases the number of modes and decreases the magnitude of the oscillations. The magnitude of the oscillations can be decreased further by increasing the number of primary nodes beyond 5. Future implementations of EQMOM will allow for this. Another possible option to decrease the magnitude of oscillations is to choose the primary quadrature solution that corresponds to a greater value of \( n_{\text{optimal}} \). Note that as discussed in Section 3.2.3, multiple values of \( n_{\text{optimal}} \) may be possible. A greater value of \( n \) will result in a smoother specific intensity. This option will be explored in future work. The maximum radiation transport speed of \( c \) is achieved by EQMOM even with one primary node. Fig. 3 shows the results at \( ct = 2 \) with 300, 600 and 1200 cells and Fig. 4 shows the results at \( ct = 4 \) with 410, 820 and 1640 cells. As time progresses, the effect of scattering increases and the oscillations begin to decay. Thus for later times, using less than 5 primary nodes may suffice.

5.2. Two-beam instability problem

The two-beam instability problem is designed to test a closure’s ability to handle multi-modal distributions. It is known, for example, that entropy-based closures yield unphysical shocks in the steady state profile [5, 26, 29]. The scattering and absorption opacities are given by \( \sigma_s = 0 \) and \( \sigma_a = 2 \) respectively. There is no external source of radiation (\( S = 0 \)). The specific intensity and the corresponding moments at \( t = 0 \) are given by

\[ I(z, \mu) = 0.00005 \quad \text{and} \quad M_p (z) = 0.00005 \left[ \frac{1 - (-1)^{p+1}}{p + 1} \right] \tag{47} \]

respectively. The left and right boundaries are located at \( z_{LB} = 0 \) and \( z_{RB} = 0.5 \) respectively. Reflective boundary condition is applied at the left boundary:

\[ I_{LB} (t) = I_{LB} (-\mu) \quad \text{and} \quad M_{p, LB} = (-1)^p M_{p, LB} . \tag{48} \]

At the right boundary, Dirichlet boundary condition with a fixed baseline specific intensity value is applied:

\[ I_{RB} (t) = 0.5 \quad \text{and} \quad M_{p, RB} = 0.5 \left[ \frac{1 - (-1)^{p+1}}{p + 1} \right] . \tag{49} \]

For this problem, results are presented with the number of primary nodes varying from 1 through 4. Fig. 5 shows the steady state results with 100, 200 and 400 cells. The semi-analytic solution is obtained from [29]. EQMOM with one primary node is not able to resolve this problem. However, on increasing the number of primary nodes, the EQMOM solution quickly approaches the semi-analytic solution. None of the results show the appearance of a shock.

5.3. Reed cell problem

The Reed cell problem [66, 8] tests a model’s ability to deal with regions having different material properties. The left and right boundaries are located at \( z_{LB} = 0 \) and \( z_{RB} = 8 \) respectively. The space between the two boundaries is divided into five regions. The opacities and the external source terms in these regions are given in Table 2. The specific intensity and the corresponding moments at \( t = 0 \) are given by \( I(z, \mu) = 0 \) and \( M_p (z) = 0 \) respectively. A reflective boundary condition is applied at the left boundary.

<table>
<thead>
<tr>
<th>Region</th>
<th>( \sigma_a )</th>
<th>( \sigma_s )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z \in (0, 2) )</td>
<td>50.0</td>
<td>0.0</td>
<td>50.0</td>
</tr>
<tr>
<td>( z \in (2, 3) )</td>
<td>5.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( z \in (3, 5) )</td>
<td>0.0001</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( z \in (5, 7) )</td>
<td>0.1</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>( z \in (7, 8) )</td>
<td>0.1</td>
<td>0.9</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Fig. 4. Solutions for the plane-source problem at $ct = 4$. 
\[ I_{LB,\ell}(\mu) = I_{LB,\ell}(-\mu) \quad \text{and} \quad M_{p,\ell B,\ell} = (-1)^\ell M_{p,\ell B,\ell}. \]  

At the right boundary, a vacuum boundary condition is applied:

\[ I_{RB,\ell}(\mu) = \begin{cases} I_{RB,\ell}(\mu), & \text{if } \mu > 0, \\ 0, & \text{otherwise}. \end{cases} \]  

\( M_{p,\ell B,\ell} \) is computed accordingly.

For this problem, results are presented with the number of primary nodes varying from 1 through 5. Fig. 6 shows the steady state results with 100, 200 and 400 cells. The reference solution is obtained from [8] which used a spherical harmonic \( P_{17} \) angular approximation and a regular spatial finite element mesh consisting of 400 elements. As the number of primary nodes is increased, the EQMOM solution approaches the reference solution. Note that for all the other regions except for the third one \((z \in (3, 5))\), the use of three primary nodes gives a fairly good solution. It is only the third region that requires a larger number of primary nodes, as is expected given the small cross-section values. With 5 primary nodes, the EQMOM solution in this region is close to the reference solution. In the current work, the same number of nodes is used in all the cells. A more efficient approach will be to use an adaptive EQMOM that automatically adjusts to smaller or larger number of primary nodes as needed. The adaptive version of EQMOM is discussed in [83].

5.4. Diffusive Marshak wave problem

The diffusive Marshak wave problem [55] is an absorption dominant problem for which the transport solution and the diffusion solution are expected to agree closely. The scattering and absorption opacities are given by \( \sigma_s = 0 \) and
Fig. 6. Solutions for the Reed cell problem at steady state.
\( \sigma_a = 300/T^3 \) respectively. There is no external source of radiation \( (S = 0) \). Initially, radiation is in equilibrium with the material. The temperature, the specific intensity and the corresponding moments at \( t = 0 \) are given by

\[
T(z) = 10^{-9}, \quad I(z, \mu) = \frac{1}{2}ac \times 10^{-36} \quad \text{and} \quad M_p(z) = \frac{1}{2}ac \times 10^{-36} \times \left[ 1 - (-1)^{p+1} \right] \left( \frac{p + 1}{p} \right) \tag{52}
\]

respectively. The heat capacity is given by \( C_v = 3 \times 10^{15} \). The left and right boundaries are located at \( z_{LB} = 0 \) and \( z_{RB} = 0.6 \) respectively. Dirichlet boundary condition is applied at the left boundary by fixing the temperature:

\[
T_{LB} = 1.0, \quad I_{LB}(\mu) = \frac{1}{2}ac \quad \text{and} \quad M_{P,LB} = \frac{1}{2}ac \left[ 1 - (-1)^{p+1} \right] \left( \frac{p + 1}{p} \right) \tag{53}
\]

At the right boundary, vacuum boundary condition is applied:

\[
I_{RB}(\mu) = \begin{cases} 
I_{RB}(\mu), & \text{if } \mu > 0, \\
0, & \text{otherwise}.
\end{cases} \tag{54}
\]

\( M_{P,RB} \) is computed accordingly.
For this problem, results are presented with the number of primary nodes equal to 1 and 2. Fig. 7 shows the results at $ct = 300, 1500$ and $3000$. The semi-analytic solution is obtained from [64]. Three successively refined grids with 30, 60 and 120 cells are used. For this diffusion limit problem, it seems that EQMOM with just one primary node is sufficient. Any increase in the number of primary nodes, does not change the solution. The EQMOM solution approaches the semi-analytic solution as the grid is refined. However, on relatively coarse grids, the wave front is not captured properly. This is the defect of the discretization scheme rather than the EQMOM. To capture the wave front, the discretization scheme should preserve the asymptotic diffusion limit [3,35,36,41,42,54,55]. The discretization scheme used here is not asymptotic preserving. However, it must be reiterated that the aim of the current paper is to demonstrate the application of EQMOM to radiation transport modeling. The development of discretization schemes for EQMOM that are asymptotic preserving will be considered in future work.

5.5. Non-equilibrium Marshak wave problem

This is also referred to as Marshak wave problem in thin medium [55]. All the parameters and initial and boundary conditions are the same as the diffusive Marshak wave problem except for the absorption opacity which is given by $\sigma_a = 3/T^2$. In this problem, the cold cells are optically thick and the warm cells are optically thin. Due to the presence of transport effects in this problem, the diffusive solution is not adequate.
For this problem, results are presented with the number of primary nodes equal to 1 and 2. The reference solution is obtained from [55] which used a spherical harmonic $P_2$ angular approximation and a regular spatial mesh consisting of 6000 cells. The reference solution used a discontinuous Galerkin method with two temperature unknowns per cell. Fig. 8 shows the EQMOM results at $ct = 30$. Three successively refined grids with 30, 60 and 120 cells are used. With just 2 primary nodes, the EQMOM solution closely resembles the reference solution.

5.6. Su–Olson problem

The Su–Olson problem [62] is a non-equilibrium radiative transfer problem. The scattering and absorption opacities are given by $\sigma_s = 0$ and $\sigma_a = 1$ respectively. The external source of radiation is given by

$$ S = \begin{cases} \alpha c, & \text{if } 0 < x < 0.5 \quad \text{and} \quad 0 \leq ct < 10, \\ 0, & \text{otherwise}. \end{cases} \quad (55) $$

Initially, radiation is in equilibrium with the material. The temperature, the specific intensity and the corresponding moments at $t = 0$ are given by

$$ T(z) = 10^{-2.5}, \quad I(z, \mu) = \frac{1}{2} \alpha c \times 10^{-10} \quad \text{and} \quad M_p(z) = \frac{1}{2} \alpha c \times 10^{-10} \times \left[ 1 - \left( \frac{-1}{p+1} \right) \right] $$

respectively. The heat capacity is given by $C_v = 4\alpha T^3$. The left and right boundaries are located at $z_{LB} = 0$ and $z_{RB} = 30$ respectively. Reflective boundary condition is applied at the left boundary:

$$ I_{LB,l}(\mu) = I_{LB,r}(-\mu) \quad \text{and} \quad M_{p,LB,l} = (-1)^p M_{p,LB,r}. $$

Dirichlet boundary condition is applied at the right boundary by fixing the temperature:

$$ T_{RB,r} = 10^{-2.5}, \quad I_{RB,r}(\mu) = \frac{1}{2} \alpha c \times 10^{-10} \quad \text{and} \quad M_{p,RB,r} = \frac{1}{2} \alpha c \times 10^{-10} \times \left[ 1 - \left( \frac{-1}{p+1} \right) \right]. $$

For this problem, results are presented with the number of primary nodes equal to 1 and 2. Fig. 9 shows the results at $ct = 1.316$ and 10. Fig. 10 shows the same results on a logarithmic scale. The semi-analytic solution is obtained from [74]. Three successively refined grids with 240, 480 and 960 cells are used. With just 2 primary nodes, the EQMOM solution closely resembles the semi-analytic solution.

6. Conclusions

The equations for radiation transport modeling have been formulated using EQMOM with special attention to realizability and a set of standard one-dimensional benchmark problems encompassing optically thin, thick and transition regimes has been solved. The results look very promising and lay the foundation for extending the same framework to multiple dimensions. For multiple dimensions, a bivariate version of EQMOM is needed. One version of QMOM in multiple dimensions called conditional quadrature method of moments (CQMOM) [84] has been developed using conditional probability. A similar approach will be used in developing multi-dimensional EQMOM. Furthermore, high-order accurate realizable spatial and temporal discretization schemes that preserve the asymptotic diffusion limit need to be developed. These tasks will be undertaken in future work.

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References