

On the Connection Between the Correction and Weighting Functions in the Correction Procedure via Reconstruction Method

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Abstract In this note, the connection between the correction and weighting functions for the correction procedure via reconstruction (CPR) method in 1D is addressed. A one-parameter family of weighting functions is constructed from the discontinuous test space. It is found that if the solution polynomials lie in the space P^k , then the first k weighting functions can always be chosen as the basis of the polynomial space P^{k-1} and the last weighting function can be selected as a piece-wise continuous polynomial. There exists at least one set of weighting functions which can recover the energy stable flux reconstruction (ESFR) schemes. This strategy has been successfully applied to recover several known high-order discontinuous schemes, including DG, SD, SV, and Huynh's g_2 scheme.

Keywords Correction procedure via reconstruction (CPR) · Energy stable flux reconstruction · Weighting function · Correction function

1 Introduction

In 2007, Huynh introduced a new nodal formulation named the flux reconstruction (FR) approach for hyperbolic conservation laws [1]. The FR approach is a discontinuous “finite difference” like differential formulation, which unifies several existing methods in the framework, including the discontinuous Galerkin (DG) [2, 3], staggered grid spectral element [4], spectral volume (SV) [5] and spectral difference (SD) [6] methods. In addition, several new schemes have been discovered, which possess interesting properties. For example, the g_2 scheme is perhaps the most efficient scheme and also allows a larger time step than the DG scheme. In the FR approach, the reconstructed flux polynomial is computed based on the correction function, and different schemes are associated with different correction functions. For example, the DG scheme is associated with the Radau polynomial, while the g_2 scheme is related to a weighted average of the Radau polynomials. In 2009, Wang and Gao [7, 8] extended the flux reconstruction approach to simplex elements under the

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lifting collocation penalty (LCP) framework. Instead of reconstructing the flux polynomial, a correction field is computed using a lifting operator based on weighting functions. Different weighting functions result in different schemes. Although in 1D the correction functions can be special polynomials such as the Legendre or Radau polynomials, on a simplex in 2D, such a relation remains an open problem. Also in 2009, Huynh [9] extended the FR framework to handle the diffusion equation, which lays the foundation for the extension to the Navier-Stokes equations. In addition, the FR and LCP methods are renamed the correction procedure via reconstruction or CPR in short. Later, Gao and Wang [10] extended the CPR method to the 2D Navier-Stokes equations on a mixed unstructured mesh and analyzed the performance of several compact approaches for the viscous term, including the BR2 and I-continuous schemes. The CPR method was further extended to the 3D Navier-Stokes equations on mixed grids by Haga, Gao and Wang [11]. More details on the CPR formulation are contained in two book chapters [12].

In 2010, a proof of energy stability for a particular class of the FR method for the 1D advection equation was provided by Vincent, Castonguay and Jameson [13]. This class of schemes is called energy stable flux reconstruction (ESFR) schemes or Vincent-Castonguay-Jameson-Huynh (VCJH) schemes. The precursor of this work is the proof of an energy stable SD scheme provided by Jameson [14]. Note that in 2007 Huynh [1] discovered a simple description for this ESFR family using the criterion of least squares. In 2011, Castonguay, Vincent and Jameson [15] further extended the ESFR approach to triangular elements. This approach is similar to the weighting function based CPR method in that the flux correction is not directly used to reconstruct the fluxes, but to reconstruct a correction field for the flux divergence. Satisfactory results have been demonstrated for both the 2D Euler [16] and Navier-Stokes equations [17]. Recently, Vincent, Castonguay and Jameson [18] summarized a series of work on the development of the ESFR scheme and showed the stability analysis based on the von Neumann analysis. The non-linear stability of the ESFR approach and a general guidance to minimize the non-linear alias errors were discussed in [19]. In addition, Huynh [20] also extended the flux reconstruction framework to the 2D triangular elements in 2011.

The present note explores the connection between the correction functions used to obtain the reconstructed flux polynomial and the weighting functions used in the weighted residual formulation to derive the correction field for the flux divergence. In addition, since the ESFR schemes already address the issue of stability, there may be a connection between weighting functions and scheme stability. As mentioned earlier, one such connection between the correction functions and the weighting functions already exists. For the 1D DG method, the correction functions are the Radau polynomials, whereas the weighting functions are the basis of the corresponding polynomial space. The question is then: Do such connections exist for other schemes? The SV method can be recovered using piece-wise constant weighting functions. Due to the equivalence of the SV and SD schemes in 1D [21], such weighting functions can also recover the SD method. In 2007, Huynh [1] discussed the space of weighting (test) functions which can cast the flux reconstruction approach in the weighted residual framework, and found that different subspaces of the polynomial space P^{k+1} can be used to serve as the weighting function space for different high-order methods with a degree k solution space. DG is the special case with the test space of P^k , which does not contain polynomials of degree $k + 1$. From a different perspective, we explore new weighting function spaces which can contain discontinuous polynomials with degree at most k in this study. Until now, the selection of weighting functions has not received much attention, and its impact on the scheme stability, CFL limit and accuracy is still not well understood. The present note attempts to shed light on these areas.

The note is organized as follows. In Sect. 2, the CPR formulation is presented. Then, a general connection between the correction and the weighting functions is established in Sect. 3. In Sect. 4, several choices of weighting functions that can recover existing high-order schemes are presented. Conclusions are summarized in Sect. 5.

2 The 1D Flux Reconstruction or CPR Formulation

Consider the 1D scalar conservation law,

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad (1)$$

on a domain Ω . The domain is partitioned into N non-overlapping elements, denoted by $\Omega_n = \{x | x_{n-1} < x < x_n\}$, $n = 1, \dots, N$. In each element, suppose the exact conservative variable u and flux f can be approximated numerically by u_n^δ and f_n^δ respectively. Then for each element, we have

$$\frac{\partial u_n^\delta}{\partial t} + \frac{\partial f_n^\delta}{\partial x} = 0. \quad (2)$$

On performing a time independent transformation of Eq. (2) from the physical domain Ω_n to the computational domain (standard element) $\Omega_s = \{\xi | \xi \in [-1, 1]\}$, the evolution of $\hat{u}^\delta$ within the standard element Ω_s reads

$$\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^\delta}{\partial \xi} = 0, \quad \text{with } \hat{u}^\delta = |J|u^\delta, \quad \hat{f}^\delta = f^\delta \text{ and } |J| = x_\xi. \quad (3)$$

In the FR approach, $\hat{u}^\delta$ and $\hat{f}^\delta$ are polynomials of degree k and $k+1$ respectively. Moreover, the flux $\hat{f}^\delta$ is expressed as $\hat{f}^\delta = \hat{f}^{\delta D} + \hat{f}^{\delta C}$. Herein, $\hat{f}^{\delta D}$, a polynomial of degree k , stands for the local flux, which is discontinuous at the cell interface, and $\hat{f}^{\delta C}$, a polynomial of degree $k+1$, stands for the flux correction over the standard element. Note that the flux correction term $\hat{f}^{\delta C}$ is used to ensure conservation, and approximate Riemann solvers, such as Rusanov, Roe and AUSM type solvers, can be used to compute the common interface fluxes (or numerical fluxes). We denote the numerical fluxes associated with the left and right interfaces of the standard element by $\hat{f}_L^{\delta I}$ and $\hat{f}_R^{\delta I}$ respectively. Note that $\hat{f}_L^{\delta I} = \hat{f}^\delta(-1, t)$ and $\hat{f}_R^{\delta I} = \hat{f}^\delta(1, t)$ in a standard element. After this, two degree $k+1$ polynomials g_L and g_R approximating zero in some sense are chosen to reconstruct $\hat{f}^{\delta C}$

$$\hat{f}^{\delta C} = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})g_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})g_R, \quad (4)$$

with

$$g_L(-1) = 1, \quad g_L(1) = 0, \quad (5)$$

$$g_R(-1) = 0, \quad g_R(1) = 1. \quad (6)$$

Thus, Eq. (3) can be rewritten as

$$\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^{\delta D}}{\partial \xi} + \frac{\partial \hat{f}^{\delta C}}{\partial \xi} = 0. \quad (7)$$

Let $\delta^C = \partial \hat{f}^{\delta C} / \partial \xi$, which is a degree k polynomial called the correction field. Then Eq. (7) becomes

$$\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^{\delta D}}{\partial \xi} + \delta^C = 0. \quad (8)$$

As suggested in [1, 13], for the reconstruction of the correction functions, the symmetry relationship $g_L(\xi) = g_R(-\xi)$ is imposed to eliminate any possible bias in the correction polynomials. Several correction polynomials have been identified to associate with known schemes. For example, as shown in [1], the correction polynomials g_{DG} for the DG scheme are the right and left Radau polynomials

$$g_{L,k+1} = R_{R,k+1} = \frac{(-1)^{k+1}}{2}(P_{k+1} - P_k), \quad g_{R,k+1} = R_{L,k+1} = \frac{1}{2}(P_{k+1} + P_k), \quad (9)$$

where P_{k+1} and P_k are Legendre polynomials of degree $k+1$ and k respectively. The correction polynomials for the energy stable SD scheme g_{Ga} are

$$g_{L,k+1} = \frac{k+1}{2k+1}R_{R,k+1} + \frac{k}{2k+1}R_{R,k}, \quad g_{R,k+1} = \frac{k+1}{2k+1}R_{L,k+1} + \frac{k}{2k+1}R_{L,k}. \quad (10)$$

The correction polynomials g_2 for Huynh's g_2 scheme are denoted by

$$g_{L,k+1} = \frac{k}{2k+1}R_{R,k+1} + \frac{k+1}{2k+1}R_{R,k}, \quad g_{R,k+1} = \frac{k}{2k+1}R_{L,k+1} + \frac{k+1}{2k+1}R_{L,k}. \quad (11)$$

All these forms are generalized in [13] as special members of an energy stable scheme family defined as

$$g_{L,k+1} = \frac{(-1)^k}{2} \left[P_k - \left(\frac{\eta_k P_{k-1} + P_{k+1}}{1 + \eta_k} \right) \right], \quad g_{R,k+1} = \frac{1}{2} \left[P_k + \left(\frac{\eta_k P_{k-1} + P_{k+1}}{1 + \eta_k} \right) \right], \quad (12)$$

where

$$\eta_k = \frac{c(2k+1)(a_k k!)^2}{2}, \quad a_k = \frac{(2k)!}{2^k (k!)^2}. \quad (13)$$

For the DG scheme, $c = 0$; for the energy stable SD scheme, $c = 2k/(2k+1)(k+1)(a_k k!)^2$; for the g_2 scheme, $c = 2(k+1)/(2k+1)k(a_k k!)^2$.

Note that a closely-related one parameter family of the correction polynomials g_L and g_R has been shown in [9, 20], which reads

$$g_{L,k+1} = \alpha_k R_{R,k+1} + (1 - \alpha_k) R_{R,k}, \quad g_{R,k+1} = \alpha_k R_{L,k+1} + (1 - \alpha_k) R_{L,k}, \quad (14)$$

with $\alpha_k \in [0, 1]$. This implies that the correction polynomials are required to be at least orthogonal to the polynomial space P^{k-2} . Also, it turns out that α_k is related to η_k as below

$$\alpha_k = \frac{1}{1 + \eta_k}. \quad (15)$$

The same formulation Eq. (7) can also be derived using the weighted approach [7, 8, 12]. The numerical solution $\hat{u}^\delta$ must satisfy the following weighted residual equation

$$\int_{-1}^1 w \frac{\partial \hat{u}^\delta}{\partial t} d\xi + \int_{-1}^1 w \frac{\partial \hat{f}^\delta}{\partial \xi} d\xi = 0. \quad (16)$$

Herein, w is the weighting function and sits in the test space. Note that the test space has the same dimension as the solution space, and is chosen in a manner to guarantee the existence and uniqueness of the numerical solution.

Applying integration by parts to Eq. (16), we obtain

$$\int_{-1}^1 w \frac{\partial \hat{u}^\delta}{\partial t} d\xi - \int_{-1}^1 \hat{f}^{\delta D} \frac{\partial w}{\partial \xi} d\xi + \hat{f}_R^{\delta I} w(1) - \hat{f}_L^{\delta I} w(-1) = 0. \quad (17)$$

On integrating Eq. (17) by part once again, the strong form reads

$$\int_{-1}^1 w \frac{\partial \hat{u}^\delta}{\partial t} d\xi + \int_{-1}^1 w \frac{\partial \hat{f}^{\delta D}}{\partial \xi} d\xi + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})w(1) - (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})w(-1) = 0. \quad (18)$$

A correction field δ^C is reconstructed according to

$$\int_{-1}^1 w \delta^C d\xi = (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})w(1) - (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})w(-1). \quad (19)$$

Thus, Eq. (18) can be written as

$$\int_{-1}^1 w \left(\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^{\delta D}}{\partial \xi} + \delta^C \right) d\xi = 0. \quad (20)$$

If the flux vector $\hat{f}^{\delta D}$ is a linear function of the state variable $\hat{u}^\delta$, and $\hat{u}^\delta$ is a degree k polynomial, then $\partial \hat{f}^{\delta D} / \partial \xi$ belongs to the space of polynomials of degree k or less (P^k). Since the test space is selected to ensure a unique solution, the following differential form holds

$$\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^{\delta D}}{\partial \xi} + \delta^C = 0. \quad (21)$$

For nonlinear conservation laws, $\partial \hat{f}^{\delta D} / \partial \xi$ is not in general an element of P^k . As a result, Eq. (20) cannot be reduced to Eq. (21). In this case, the most obvious choice is to project $\partial \hat{f}^{\delta D} / \partial \xi$ onto P^k . Denote a projection of $\partial \hat{f}^{\delta D} / \partial \xi$ to P^k by $\Pi(\partial \hat{f}^{\delta D} / \partial \xi)$. Then Eq. (20) reduces to

$$\frac{\partial \hat{u}^\delta}{\partial t} + \Pi \left(\frac{\partial \hat{f}^{\delta D}}{\partial \xi} \right) + \delta^C = 0. \quad (22)$$

Two efficient approaches to calculate $\Pi(\partial \hat{f}^{\delta D} / \partial \xi)$ are presented in [7, 8, 12]. One is called the LP (Lagrange polynomial), and the other called the CR (chain-rule) approach.

Since the same formulation can be derived from both the FR and LCP approaches, we have decided to unify them under the name CPR. From here on, FR, LCP and CPR will mean the same thing. Next we focus on finding the connection between the correction function g_L , g_R , and the weighting function w .

3 Connection Between the Correction and the Weighting Functions

It is obvious from the last section that a CPR scheme depends on the following factors assuming the solution space is P^k :

- (1) The Riemann solver,
- (2) The way the divergence is computed (LP or CR), and
- (3) The correction function or the weighting function.

Assuming the first two factors are the same, the correction function or the weighting function determines the scheme. For example, in the DG scheme, the correction function is the Radau polynomial, while the weighting function is the basis of P^k [8]. Next we briefly review this result.

3.1 The Correction and the Weighting Functions for the DG Scheme

The right and left Radau polynomials of degree $k + 1$ in Eq. (9) are used to build the DG scheme. According to Eqs. (4) and (7), it is clear that

$$\delta^C = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})R'_R + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})R'_L. \quad (23)$$

On using the following identities [22] of the Legendre polynomials,

$$P'_i = P'_{i-2} + [2(i-1) + 1]P_{i-1}, \quad (24)$$

and

$$P'_0 = 0, \quad P'_1 = P_0, \quad (25)$$

it is easy to verify that

$$(-1)^{k+1}(P'_{k+1} - P'_k) = \sum_{i=0}^k (-1)^{i+1}(2i+1)P_i, \quad (26)$$

$$P'_{k+1} + P'_k = \sum_{i=0}^k (2i+1)P_i. \quad (27)$$

Substituting Eq. (9) into Eq. (23) and using Eqs. (26) and (27), we obtain

$$\begin{aligned} \delta^C &= (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D}) \frac{(-1)^{k+1}}{2} (P'_{k+1} - P'_k) + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) \frac{1}{2} (P'_{k+1} + P'_k) \\ &= \sum_{i=0}^k \frac{2i+1}{2} [(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})(-1)^{i+1} + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})] P_i. \end{aligned} \quad (28)$$

Since δ^C in Eq. (21) or (22) is a polynomial of degree k , we can expanding δ^C as a summation of Legendre polynomials

$$\delta^C = \sum_{i=0}^k \psi_i P_i, \quad (29)$$

where ψ_i are the expansion coefficients. Note that $\{P_0, P_1, \dots, P_k\}$ is a complete orthogonal basis of P^k and any complete basis of P^k can be expressed with a linear combination of $\{P_0, P_1, \dots, P_k\}$. Then choose the weighting functions from the same space as $\{P_0, P_1, \dots, P_k\}$. Note that other basis of P^k should also work for the purpose of constructing weighting functions.

Substituting Eq. (29) and weighting functions $\{P_0, P_1, \dots, P_k\}$ into Eq. (19), a linear system governing the expansion coefficients can be formulated as follows

$$\begin{pmatrix} \int_{-1}^1 P_0 P_0 d\xi & \cdots & \int_{-1}^1 P_0 P_i d\xi & \cdots & \int_{-1}^1 P_0 P_k d\xi \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{-1}^1 P_0 P_i d\xi & \cdots & \int_{-1}^1 P_i P_i d\xi & \cdots & \int_{-1}^1 P_i P_k d\xi \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{-1}^1 P_0 P_k d\xi & \cdots & \int_{-1}^1 P_i P_k d\xi & \cdots & \int_{-1}^1 P_k P_k d\xi \end{pmatrix} \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_i \\ \vdots \\ \psi_k \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_i \\ \vdots \\ b_k \end{pmatrix}, \quad (30)$$

where $b_i = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})(-1)^{i+1} + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})$.

By applying the following properties of the Legendre polynomial,

$$\int_{-1}^1 P_i P_j d\xi = \begin{cases} 0, & \text{if } i \neq j \\ \frac{2}{2i+1}, & \text{if } i = j \end{cases}, \quad (31)$$

$$P_i(1) = 1, \quad P_i(-1) = (-1)^i. \quad (32)$$

we obtain,

$$\begin{pmatrix} \frac{2}{1} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{2}{2i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & \frac{2}{2k+1} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_i \\ \vdots \\ \psi_k \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_i \\ \vdots \\ b_k \end{pmatrix}. \quad (33)$$

Therefore,

$$(\psi_0, \dots, \psi_i, \dots, \psi_k)^T = \left(\frac{b_0}{2}, \dots, \frac{(2i+1)b_i}{2}, \dots, \frac{(2k+1)b_k}{2} \right)^T, \quad (34)$$

and

$$\delta^C = \sum_{i=0}^k \frac{2i+1}{2} [(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})(-1)^{i+1} + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})] P_i. \quad (35)$$

Equation (35) is exactly the same as Eq. (28), indicating that the correction functions of degree $k+1$ Radau polynomials result in the same scheme as weighting functions of Legendre polynomials $\{P_0, P_1, \dots, P_k\}$ does.

3.2 Connection Between the Correction and the Weighting Functions for Energy Stable FR Schemes

Next we attempt to establish a general connection between the correction function and the weighting function for the ESFR schemes developed in [13]. In the present study, we limit the weighting functions to be continuous and discontinuous polynomials on $[-1, 1]$. Note that other types of functions, such as trigonometric and exponential functions, etc. might also work, but these choices will not be explored here.

Following Eq. (29), a degree k correction field δ^C can be expanded with respect to the Legendre polynomials. By revisiting Eq. (19), we found that δ^C can also be written as

$$\delta^C = (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\delta_{R,k} + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\delta_{L,k}, \quad (36)$$

where $\delta_{R,k} \in P^k$, $\delta_{L,k} \in P^k$ and $\delta_{R,k} = \sum_{i=0}^k a_{i,R} P_i$, $\delta_{L,k} = \sum_{i=0}^k a_{i,L} P_i$.

By comparing Eq. (36) with (29), it is clear that

$$\psi_i = a_{i,R}(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) + a_{i,L}(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D}). \quad (37)$$

Substituting Eq. (36) into Eq. (19), we obtain

$$\int_{-1}^1 w \delta_{R,k} d\xi = w(1), \quad \int_{-1}^1 w \delta_{L,k} d\xi = -w(-1). \quad (38)$$

It is observed that δ_R and δ_L defined above are closely related to the correction functions proposed in [1, 13] in the following sense

$$\delta_{R,k}(\xi) = \frac{dg_{R,k+1}(\xi)}{d\xi} = g'_{R,k+1}, \quad \delta_{L,k}(\xi) = \frac{dg_{L,k+1}(\xi)}{d\xi} = g'_{L,k+1}, \quad (39)$$

where $g_{R,k+1}$ and $g_{L,k+1}$ are the correction functions in the ESFR schemes as given in Eq. (12).

Using Eqs. (24) and (25), it is clear that

$$\begin{aligned} g'_R &= \frac{1}{2} \left[P'_k + \left(\frac{\eta_k P'_{k-1} + P'_{k+1}}{1 + \eta_k} \right) \right] \\ &= \frac{1}{2} \left(P'_k + P'_{k-1} + \frac{2k+1}{1 + \eta_k} P_k \right) \\ &= \frac{1}{2} \left(\sum_{j=0}^{k-1} (2j+1) P_j + \frac{2k+1}{1 + \eta_k} P_k \right). \end{aligned} \quad (40)$$

Similarly,

$$g'_L = \frac{1}{2} \left(\sum_{j=0}^{k-1} (2j+1) P_j (-1)^{j+1} + \frac{2k+1}{1 + \eta_k} P_k (-1)^{k+1} \right). \quad (41)$$

Note that in order to build a degree k correction field δ^C , as shown in Eq. (29) or (36), $k+1$ weighting functions which simultaneously satisfy the two equations in Eq. (38) are needed. Using the properties of Legendre polynomials (i.e. Eqs. (31) and (32)), we can verify that if the first k weighting functions are chosen as the first k Legendre polynomials P_i , $i = 0, \dots, k-1$, Eq. (38) is always satisfied. Next we examine how to find the last weighting function w_k that satisfies Eq. (38) with $\delta_{R,k}$ and $\delta_{L,k}$ defined by Eqs. (40) and (41) respectively.

As mentioned before, we assume the weighting functions to be arbitrary piece-wise continuous polynomials on $[-1, 1]$. It is obvious that a weighting function of a continuous degree k polynomial will yield the correction field for the DG scheme. In order to build different high-order schemes, an intuitive and simple choice is to choose a discontinuous polynomial with one discontinuity at $\xi = 0$. However, we failed to establish a connection

between the correction and weighting functions at all orders of accuracy with such a choice. Therefore, we consider the following function with two symmetric discontinuities,

$$w_k = \begin{cases} P_k, & -1 \leq \xi \leq -\xi_C, \\ \sum_{i=0}^k \alpha_i P_i, & -\xi_C \leq \xi \leq \xi_C, \\ P_k, & \xi_C \leq \xi \leq 1, \end{cases} \quad (42)$$

where $\xi_C \in (0, 1)$.

Substituting Eq. (42) into Eq. (38), we obtain the following equations

$$\begin{aligned} \sum_{j=0}^k \int_{-\xi_C}^{\xi_C} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_j d\xi \alpha_j &= 1 - a_{k,R} \frac{2}{2k+1} + \int_{-\xi_C}^{\xi_C} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_k d\xi, \\ \sum_{j=0}^k \int_{-\xi_C}^{\xi_C} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_j d\xi \alpha_j &= (-1)^{k+1} - a_{k,L} \frac{2}{2k+1} + \int_{-\xi_C}^{\xi_C} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_k d\xi. \end{aligned} \quad (43)$$

From Eqs. (40) and (41), it is found that $a_{i,R} = \frac{2i+1}{2}$, $a_{i,L} = \frac{2i+1}{2}(-1)^{i+1}$, $i = 0, \dots, k-1$ and $a_{k,R} = \frac{2k+1}{2(1+\eta_k)}$, $a_{k,L} = \frac{2k+1}{2(1+\eta_k)}(-1)^{k+1}$. A linear system governing $\{\alpha_0, \alpha_1, \dots, \alpha_k\}$ needs to be solved to determine the last weighting function w_k . Now let's inspect the property of the linear system.

If $k = 0$, the system is over determined. But a further examination reveals that the two equations reduce to one

$$\int_{-\xi_C}^{\xi_C} \frac{P_0 P_0}{2(1+\eta_0)} d\xi \alpha_0 = 1 - \frac{1}{1+\eta_0} + \int_{-\xi_C}^{\xi_C} \frac{P_0 P_0}{2(1+\eta_0)} d\xi. \quad (44)$$

If $k = 1$, Eq. (43) has the following unique solutions

$$\alpha_0 = 0, \quad \alpha_1 = \frac{\eta_1 + \xi_C^3}{\xi_C^3}. \quad (45)$$

If $k > 1$, then the system is under determined and $k-1$ more conditions are needed to close the system, which clearly indicates the last weighting function is not unique. These conditions can be flexible to some extent and may be chosen based on the type of system derived above. For example, let $\alpha_i = 0$, $i = 0, \dots, k-2$. Then a 2×2 system is solved to obtain α_{k-1} and α_k . This completes the procedure to find all the weighting functions for the particular ESFR schemes.

Next we explore the simplest last weighting function possible in the form of Eq. (42). To achieve that, let $\alpha_i = 0$, $i = 0, \dots, k-2$. Then the remaining system about α_{k-1} and α_k can be written as

$$\begin{aligned} C_{k-1,R} \alpha_{k-1} + C_{k,R} \alpha_k &= 1 - a_{k,R} \frac{2}{2k+1} + C_{k,R}, \\ C_{k-1,L} \alpha_{k-1} + C_{k,L} \alpha_k &= (-1)^{k+1} - a_{k,L} \frac{2}{2k+1} + C_{k,L}, \end{aligned} \quad (46)$$

where

$$\begin{aligned}
 C_{k-1,R} &= \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_{k-1} d\xi, & C_{k,R} &= \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_k d\xi, \\
 C_{k-1,L} &= \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_{k-1} d\xi, & C_{k,L} &= \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_k d\xi, \\
 a_{i,R} &= \frac{2i+1}{2}, & a_{i,L} &= \frac{2i+1}{2}(-1)^{i+1}, & \text{for } i &= 0, \dots, k-1, \\
 a_{k,R} &= \frac{2k+1}{2(1+\eta_k)}, & a_{k,L} &= \frac{2k+1}{2(1+\eta_k)}(-1)^{k+1}.
 \end{aligned} \tag{47}$$

In order to further gain insights into the above linear system (Eq. (46)), we take advantage of the following properties.

If k is odd, $k-1$ and $k+1$ are even, then the following relations hold.

(1) $a_{k,R} = a_{k,L}$

(2) $C_{k-1,L} = -C_{k-1,R}$ and $C_{k,L} = C_{k,R}$. Specifically, we have

$$\begin{aligned}
 \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_{k-1} d\xi &= - \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_{k-1} d\xi, \\
 \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_k d\xi &= \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_k d\xi.
 \end{aligned} \tag{48}$$

If k is even, $k-1$ and $k+1$ are odd, then the following relations hold.

(1) $a_{k,R} = -a_{k,L}$

(2) $C_{k-1,L} = C_{k-1,R}$ and $C_{k,L} = -C_{k,R}$. Specifically, we have

$$\begin{aligned}
 \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_{k-1} d\xi &= \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_{k-1} d\xi, \\
 \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,L} P_i \right) P_k d\xi &= - \int_{-\xi_c}^{\xi_c} \left(\sum_{i=0}^k a_{i,R} P_i \right) P_k d\xi.
 \end{aligned} \tag{49}$$

It follows that if k is odd, then

$$1 - a_{k,R} \frac{2}{2k+1} + C_{k,R} = (-1)^{k+1} - a_{k,L} \frac{2}{2k+1} + C_{k,L}; \tag{50}$$

if k is even, then

$$- \left(1 - a_{k,R} \frac{2}{2k+1} + C_{k,R} \right) = (-1)^{k+1} - a_{k,L} \frac{2}{2k+1} + C_{k,L}. \tag{51}$$

Based on the above discussions, no matter whether k is odd or even, the solutions of Eq. (46) are

$$\alpha_{k-1} = 0, \quad \alpha_k = 1 + \frac{\eta_k}{(1+\eta_k)C_{k,R}}. \tag{52}$$

Therefore, Eq. (42) can be written as

$$w_k = \begin{cases} P_k, & -1 \leq \xi \leq -\xi_C, \\ \alpha_k P_k, & -\xi_C \leq \xi \leq \xi_C, \\ P_k, & \xi_C \leq \xi \leq 1, \end{cases} \quad (53)$$

where $\xi_C \in (0, 1)$.

Using the similar approaches aforementioned, we find a more concise way to establish the last weighting function as

$$w_k = \begin{cases} (-1)^k, & -1 \leq \xi \leq -\xi_C, \\ \alpha_k \xi^k, & -\xi_C \leq \xi \leq \xi_C, \\ 1, & \xi_C \leq \xi \leq 1, \end{cases} \quad (54)$$

where $\xi_C \in (0, 1)$ and

$$\alpha_k = \frac{\eta_k C_3 + C_4}{\eta_k C_1 + C_2}, \quad (55)$$

with

$$\begin{aligned} C_1 &= \sum_{i=0}^{k-1} \int_{-\xi_C}^{\xi_C} \frac{2i+1}{2} P_i \xi^k d\xi, & C_2 &= \sum_{i=0}^k \int_{-\xi_C}^{\xi_C} \frac{2i+1}{2} P_i \xi^k d\xi, \\ C_3 &= 1 - \sum_{i=0}^{k-1} \int_{-1}^{-\xi_C} (-1)^k \frac{2i+1}{2} P_i d\xi - \sum_{i=0}^{k-1} \int_{\xi_C}^1 \frac{2i+1}{2} P_i d\xi, \\ C_4 &= 1 - \sum_{i=0}^k \int_{-1}^{-\xi_C} (-1)^k \frac{2i+1}{2} P_i d\xi - \sum_{i=0}^k \int_{\xi_C}^1 \frac{2i+1}{2} P_i d\xi. \end{aligned} \quad (56)$$

Two test function sets for the energy stable SD scheme are plotted in Fig. 1 based on the weighting function selection procedures. The last weighting function is chosen according to

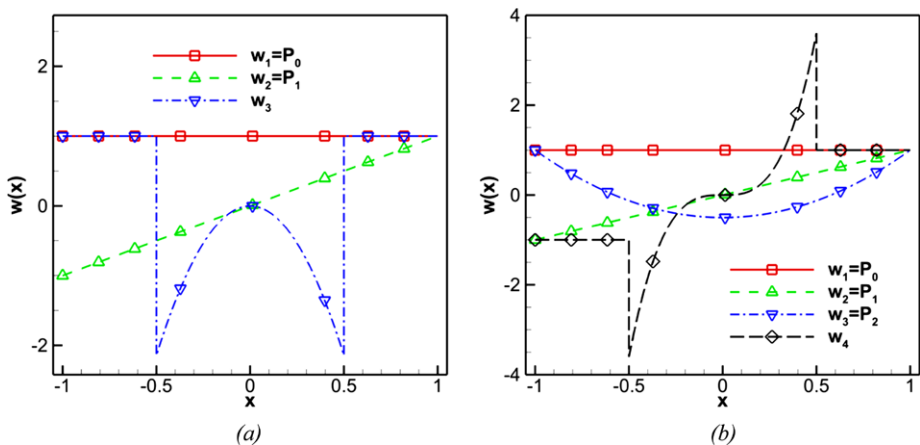


Fig. 1 The basis of test spaces for both P^2 (a) and P^3 (b) solution representations. The form of the last weighting function is selected as Eq. (54)

Table 1 The coefficient α_k of Eq. (53) and the corresponding C_1 and C_2 in Eq. (61) that can ensure the linear stability for $0 \leq k \leq 5$ with $\xi_C = 0.5$

k	C_1	C_2	α_k
0	0	$\frac{1}{2}$	$\alpha_k > -1$
1	0	$\frac{1}{8}$	$\alpha_k > -7$
2	$-\frac{3}{16}$	$\frac{25}{128}$	$\alpha_k > -\frac{79}{49}$ or $\alpha_k < -\frac{13}{3}$
3	$-\frac{9}{64}$	$\frac{121}{512}$	$\alpha_k > -\frac{319}{193}$ or $\alpha_k < -\frac{55}{9}$
4	$-\frac{225}{2048}$	$\frac{4969}{32768}$	$\alpha_k > -\frac{24199}{8569}$ or $\alpha_k < -\frac{1823}{225}$
5	$-\frac{1455}{8192}$	$\frac{23809}{131072}$	$\alpha_k > -\frac{83983}{47089}$ or $\alpha_k < -\frac{6737}{1455}$

Eq. (54) with $\xi_C = 0.5$ and the basis of test spaces for both P^2 and P^3 solution representations are shown.

Note that the weighting functions obtained from the general procedures developed above can ensure the linear stability of the 1D scalar conservation law with periodic boundary conditions, as proved in [13] for the ESFR scheme under Sobolev-type norms. A brief explanation is given next. In [13], it was shown that the scheme is stable if c in Eq. (13) satisfies the inequality

$$\frac{-2}{(2k+1)(\alpha_k k!)^2} < c < \infty. \quad (57)$$

Substituting the inequality (57) into Eq. (13), we have

$$-1 < \eta_k < \infty. \quad (58)$$

Since the selection of the first k weighting functions does not depend on η_k , the stability of the recovered scheme will only depend on the choice of the coefficient α_k in the last weighting function. In order to find the stability criteria for α_k in Eq. (53), first rearrange Eq. (52) as

$$\eta_k = \frac{C_2(\alpha_k - 1)}{1 - C_1(\alpha_k - 1)}, \quad (59)$$

where

$$C_1 = \sum_{i=0}^{k-1} \int_{-\xi_C}^{\xi_C} \frac{2i+1}{2} P_i P_k d\xi, \quad C_2 = \sum_{i=0}^k \int_{-\xi_C}^{\xi_C} \frac{2i+1}{2} P_i P_k d\xi. \quad (60)$$

Substituting Eq. (59) into the inequality (58), we obtain

$$-1 < \frac{C_2(\alpha_k - 1)}{1 - C_1(\alpha_k - 1)} < \infty. \quad (61)$$

The range for α_k with $\xi_C = 0.5$ for $0 \leq k \leq 5$ is tabulated in Table 1.

A similar rearrangement of Eq. (55) can be used to find the stability criteria for α_k in Eq. (54), and the final form of the inequality reads

$$-1 < \frac{C_4 - \alpha_k C_2}{\alpha_k C_1 - C_3} < \infty. \quad (62)$$

The range for α_k with $\xi_C = 0.5$ for $0 \leq k \leq 5$ is tabulated in Table 2.

Table 2 The coefficient α_k of Eq. (54) and the corresponding C_1, C_2, C_3 and C_4 in Eq. (62) that can ensure the linear stability for $0 \leq k \leq 5$ with $\xi_C = 0.5$

k	C_1	C_2	C_3	C_4	α_k
0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\alpha_k > -1$
1	0	$\frac{1}{8}$	1	$-\frac{1}{8}$	$\alpha_k > -9$
2	$\frac{1}{24}$	$-\frac{1}{64}$	$\frac{1}{2}$	$-\frac{7}{16}$	$\alpha_k > \frac{180}{11}$ or $\alpha_k < 12$
3	$\frac{3}{160}$	$-\frac{7}{256}$	$-\frac{1}{8}$	$-\frac{37}{128}$	$\alpha_k > \frac{210}{59}$ or $\alpha_k < -\frac{20}{3}$
4	$-\frac{9}{8960}$	$-\frac{37}{4096}$	$-\frac{7}{16}$	$\frac{23}{256}$	$-\frac{75600}{1151} < \alpha_k < \frac{3920}{9}$
5	$-\frac{295}{64512}$	$\frac{23}{16384}$	$-\frac{37}{128}$	$\frac{331}{1024}$	$\alpha_k > \frac{632016}{6169}$ or $\alpha_k < \frac{18648}{295}$

4 Recovery of Existing Methods

In the following discussions, the first k weighting functions are the first k Legendre polynomials, unless otherwise specified.

4.1 Discontinuous Galerkin Method

Use Eq. (53) as the last weighting function, and set $\alpha_i = 0$, $i = 0, \dots, k-1$, and $\alpha_k = 1$, namely,

$$w_k = P_k.$$

Then the coefficients for the correction function are calculated from Eq. (19) as

$$\psi_i = \frac{2i+1}{2} [(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) + (-1)^{i+1}(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})], \quad i = 0, \dots, k. \quad (63)$$

These coefficients are those for the DG scheme presented in [1] and [13].

Interestingly, it is found that if Eq. (54) is chosen as the last weighting function, the DG scheme can also be recovered. The corresponding weighting functions with $\xi_C = 0.5$ for $0 \leq k \leq 5$ are plotted in Fig. 2 and the coefficients α_i , $0 \leq i \leq k$ are shown in Table 3.

4.2 Spectral Difference Method

Approach 1 Use Eq. (53) as the last weighting function and set $\eta_k = k/(k+1)$. The corresponding weighting functions with $\xi_C = 0.5$ for $0 \leq k \leq 5$ are plotted in Fig. 3(a) and the coefficients α_i , $0 \leq i \leq k$ are presented in Table 4.

Correspondingly, the coefficients for the correction field are obtained as

$$\begin{aligned} \psi_i &= \frac{2i+1}{2} [(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) + (-1)^{i+1}(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})], \quad i = 0, \dots, k-1; \\ \psi_k &= \frac{k+1}{2} [(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) + (-1)^{k+1}(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})]. \end{aligned} \quad (64)$$

The recovered scheme is the only SD scheme proved to be linearly stable by Jameson [14].

Fig. 2 The last weighting functions in the form of Eq. (54) that can recover the DG method for $0 \leq k \leq 5$ with $\xi_C = 0.5$

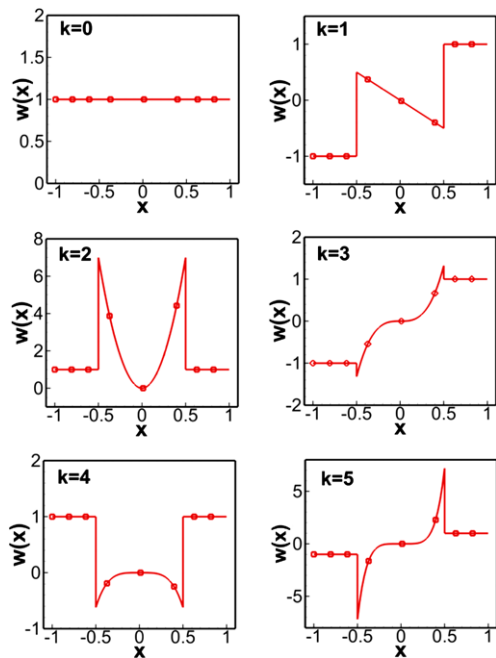


Table 3 The coefficients α_i of Eq. (54) that can recover the DG's correction field for $0 \leq k \leq 5$ with $\xi_C = 0.5$

k	α_0	α_1	α_2	α_3	α_4	α_5
0	1	–	–	–	–	–
1	0	–1	–	–	–	–
2	0	0	28	–	–	–
3	0	0	0	10.57142857142857	–	–
4	0	0	0	0	–9.94594594594595	–
5	0	0	0	0	0	230.260869565217

Table 4 The coefficients α_i of Eq. (53) that can recover the SD's correction field for $0 \leq k \leq 5$ with $\xi_C = 0.5$

k	α_0	α_1	α_2	α_3	α_4	α_5
0	1	–	–	–	–	–
1	0	5	–	–	–	–
2	0	0	10.48148148148148	–	–	–
3	0	0	0	6.73134328358209	–	–
4	0	0	0	0	13.54877932024891	–
5	0	0	0	0	0	25.77356921448551

Approach 2 Use Eq. (54) as the last weighting function and set $\eta_k = k/(k+1)$. The corresponding weighting functions with $\xi_C = 0.5$ for $0 \leq k \leq 5$ are plotted in Fig. 3(b) and the coefficients α_i , $0 \leq i \leq k$ are shown in Table 5.

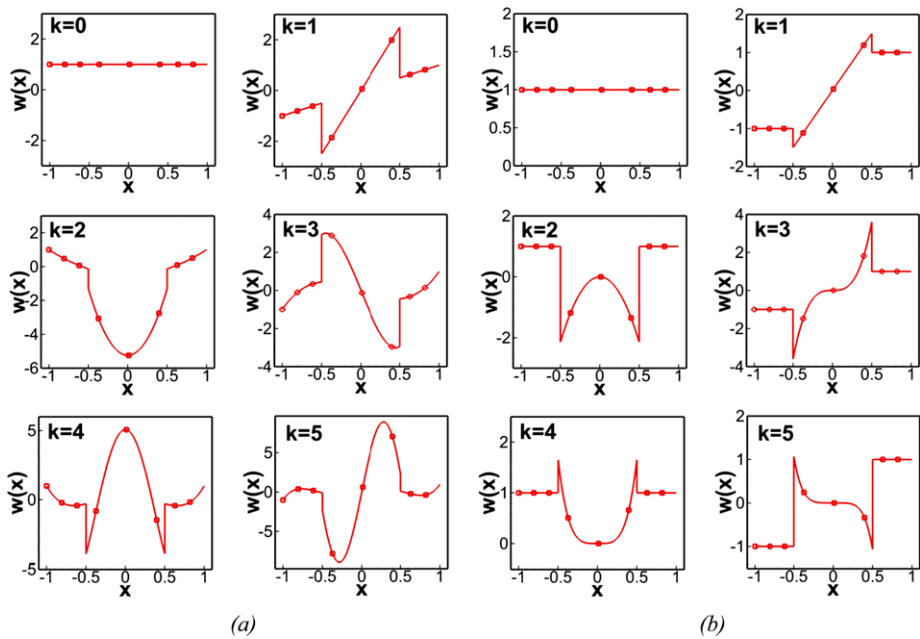


Fig. 3 The last weighting functions that can recover the SD's correction field for $0 \leq k \leq 5$ with $\xi_C = 0.5$. Specifically, (a) is drawn from Eq. (53); (b) is drawn from Eq. (54)

Table 5 The coefficients α_i of Eq. (54) that can recover the SD's correction field for $0 \leq k \leq 5$ with $\xi_C = 0.5$

k	α_0	α_1	α_2	α_3	α_4	α_5
0	1	–	–	–	–	–
1	0	3	–	–	–	–
2	0	0	–8.57142857142857	–	–	–
3	0	0	0	28.82352941176470	–	–
4	0	0	0	0	26.44731243795207	–
5	0	0	0	0	0	–34.21763048436873

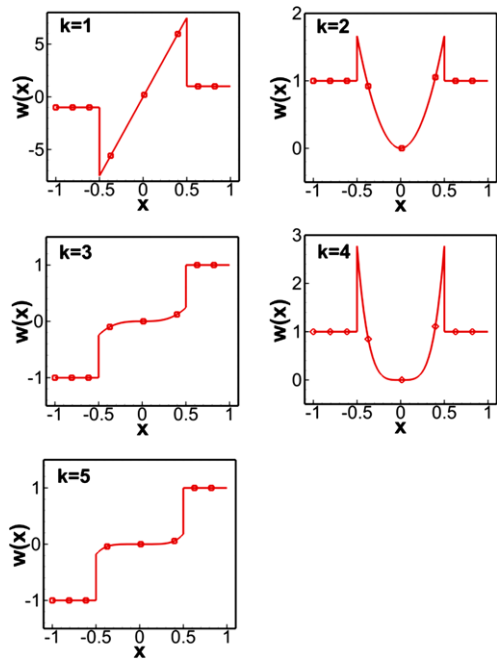
Approach 3 A special set of weighting functions can be found to recover the SD scheme. This is based on the analyses by van den Abeele, Lacor and Wang [21] that the spectral difference method is identical to the spectral volume method under certain criteria in 1D (for details please refer to [21]). Following this line of thought, the $k + 1$ weighting functions can be chosen as follows.

For $i = 0, \dots, k$,

$$w_i = \begin{cases} 1, & \xi_{C_i} \leq \xi \leq \xi_{C_{i+1}}, \\ 0, & \text{otherwise,} \end{cases} \quad (65)$$

where ξ_{C_i} and $\xi_{C_{i+1}}$ are the adjacent Legendre-Gauss points plus two end points -1 and 1 .

Fig. 4 The last weighting functions in the form of Eq. (54) that can recover the Huynh's g_2 scheme for $1 \leq k \leq 5$ with $\xi_C = 0.5$



Approach 4 Interestingly, it is found that if the first k weighting functions are chosen as the first k Legendre polynomials as

$$w_i = P_i, \quad i = 0, \dots, k-1$$

and the last weighting function is specified as

$$w_k = \begin{cases} 1, & \xi_{C_1} \leq \xi \leq \xi_{C_2}, \\ 0, & \text{otherwise,} \end{cases} \quad (66)$$

where ξ_{C_1} , ξ_{C_2} are any adjacent Legendre-Gauss points plus two end points -1 and 1 , then the SD scheme can also be recovered.

4.3 Huynh's g_2 Scheme

Use Eq. (54) as the last weighting function and set $\eta_k = (k+1)/k$. The corresponding weighting functions with $\xi_C = 0.5$ for $1 \leq k \leq 5$ are plotted in Fig. 4 and the coefficients α_i , $0 \leq i \leq k$ are presented in Table 6.

Correspondingly, the coefficients for the correction field are

$$\begin{aligned} \psi_i &= \frac{2i+1}{2} [(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) + (-1)^{i+1}(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})], \quad i = 0, \dots, k-1; \\ \psi_k &= \frac{k}{2} [(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) + (-1)^{k+1}(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})], \quad k \geq 1. \end{aligned} \quad (67)$$

Table 6 The coefficients α_i of Eq. (54) that can recover the Huynh's g_2 correction field for $1 \leq k \leq 5$ with $\xi_C = 0.5$

k	α_0	α_1	α_2	α_3	α_4	α_5
1	0	15	–	–	–	–
2	0	0	6.666666666666667	–	–	–
3	0	0	0	1.944444444444444	–	–
4	0	0	0	0	44.42033898305085	–
5	0	0	0	0	0	5.78733096085409

5 Conclusions

In this note, the connection between the correction and weighting functions for the CPR method in 1D is addressed. A one-parameter family of weighting functions is constructed from the discontinuous test space. It is found that if the solution polynomials lie in the space P^k , then the first k weighting functions can always be chosen as the basis of the polynomial space P^{k-1} and the last weighting function can be selected as a piece-wise continuous polynomial. There exists at least one set of weighting functions which can recover the energy stable FR schemes. This strategy has been successfully applied to recover several known high-order discontinuous schemes, including DG, SD, SV, and Huynh's g_2 scheme. In the future, we will attempt to find a similar connection for simplexes.

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