

On the accuracy and efficiency of discontinuous Galerkin, spectral difference and correction procedure via reconstruction methods

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ABSTRACT

Numerical accuracy and efficiency of several discontinuous high-order methods, including the quadrature-based discontinuous Galerkin (QDG), nodal discontinuous Galerkin (NDG), spectral difference (SD) and flux reconstruction/correction procedure via reconstruction (FR/CPR), for the conservation laws are analyzed and compared on both linear and curved quadrilateral elements. On linear elements, all the above schemes are one-dimensional in each natural coordinate direction. However, on curved elements, not all schemes can be reduced to a one-dimensional form, although the SD and CPR formulations remain one-dimensional by design. The efficiency and accuracy of various formulations are compared on highly skewed curved elements. Several benchmark problems are simulated to further evaluate the performance of these schemes.

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1. Introduction

Adaptive high-order (order of accuracy ≥ 3) methods have received considerable interest in the computational fluid dynamics (CFD) community recently due to their potential of delivering higher accuracy with lower computational cost than the low-order methods for problems involving complex physics and geometry, such as aero-acoustic noise prediction and vortex dominated flows. Readers are referred to several books [1–3] and reviews on these methods [4,5].

The most popular high-order method for compressible flow simulations is arguably the discontinuous Galerkin (DG) method. It was first introduced by Reed and Hill [6] for neutron transport equations in 1973 and then developed by Cockburn and Shu [7,8] and Cockburn et al. [9] for hyperbolic conservation laws. Bassi and Rebay [10,11] applied the DG discretization to the Euler and Navier–Stokes equations, and emphasized the importance of the proper treatment of the curved boundaries. Some comprehensive reviews on the DG development for both hyperbolic and elliptic problems can be found in [12,13]. Depending on how the degrees-of-freedom (DOFs) are chosen, various DG implementations have different numerical properties and efficiencies. In the most straightforward implementation of the DG approach, the Gauss quadrature is used to compute the surface and volume integrals in the weak formulation. We will call this DG implementation quadrature-based DG or QDG. A more efficient implementation, the nodal DG (NDG), was developed by Hesthaven and Warburton [2]. The NDG approach avoids the use of quadrature by expanding the (nonlinear) flux on the same Lagrange basis employed by the solution. This approach was also explored in the quadrature-free implementation of DG by Atkins and Shu [14]. Generally speaking, QDG is almost always more expensive than NDG due to the numerical quadrature and extra interpolation. For linear conservation laws, QDG and NDG are identical in accuracy. For nonlinear conservation laws, QDG is

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more accurate as NDG may run into problems associated with aliasing errors. Note that the DG methods on quadrilateral elements are also referred as DG spectral element methods [15–18].

In order to avoid the explicit numerical integrals, some finite-difference-like discontinuous formulations have been developed. One successful high-order approach of this kind is the staggered-grid (SG) multi-domain spectral method [19] or the spectral difference (SD) method [20,21]. This method directly reconstructs a flux polynomial based on fluxes on a given nodal set called flux points. Then the derivatives of the flux polynomial are used to update the solutions at the solution points. On quadrilateral or hexahedral elements, the SD method is one-dimensional by design, even for high-order elements. This is significant, as the computational cost is much less than that of the DG approaches in multiple dimensions. Later, it was found that the computational cost of SD can be further decreased as Huynh [22] and Van den Abeele et al. [23] confirmed that the SD method only depends on the location of the flux points, and the staggered-grid configuration, i.e. staggered distribution of solution points and flux points, is not necessary. This suggests that computation efficiency can be further improved if the solution points coincide with the flux points [23].

In 2007, Huynh [22] introduced the flux reconstruction (FR) approach for one-dimensional conservation law. The FR approach shares with SD the philosophy of solving the differential form of the equation. The key difference, however, is that FR allows the flux polynomial to be reconstructed by a procedure significantly more general than the interpolation procedure used by SD. Depending on how the flux polynomial is defined, the FR approach unifies many existing methods such as the DG, SD and SV [24] methods. Moreover, several new schemes with favorable properties are discovered in the FR family. Later, Huynh [25] extended the FR framework to handle the diffusion equation, which laid the foundation for the extension to the Navier–Stokes equations. In 2009, Wang and Gao [26,27] extended the flux reconstruction approach to simplex elements under the lifting collocation penalty (LCP) framework. Since the FR and LCP approaches result in the same final formulation, Huynh and Wang renamed FR and LCP schemes to ‘correction procedure via reconstruction’ or CPR schemes. Further work on the FR/CPR method can be found in [28,29,3,30–33]. A recently developed sparse line-based DG [34] on quadrilateral elements is closely related to these formulations.

In the present study, we evaluate the efficiency, accuracy and robustness of the QDG, NDG, SD and the CPR formulations on quadrilateral elements. Comparison of these formulations on triangular and other types of elements remains to be carried out. Here, accuracy and efficiency are tightly coupled. We measure the relative efficiency based on a given error threshold, following the practice used in the 1st International Workshop on High-Order CFD Methods [35].

The paper is organized as follows. For completeness, we briefly review DG, SD and CPR methods on linear elements in Section 2. Analyses of several schemes on curved elements and the modification of these schemes to enhance accuracy and efficiency are presented in Section 3. In Section 4, the implementation procedures for DG, SD and CPR methods are given and some important assumptions for different implementations are clarified. The comparisons of the algorithm accuracy and computational efficiency for the linear and Euler equations on both linear and high-order elements are presented in Section 5. Finally, conclusions are summarized in Section 6.

2. Reviews of discontinuous high-order methods on linear elements

Consider the following conservation law,

$$\frac{\partial Q}{\partial t} + \nabla \cdot \mathbf{F}(Q) = 0 \quad (1)$$

defined on $\Omega \times [0, T]$ with spatial domain Ω bounded by $\partial\Omega$, where Q is the vector of conservative variables, and $\mathbf{F} = (f, g)$ is the flux vector, which can be a linear or nonlinear function of Q .

To achieve an efficient implementation, the conservation law is usually transformed from the physical domain (x, y) into the computational domain (ξ, η) . Define the area vectors

$$\mathbf{a}^1 = (y_\eta, -x_\eta), \quad \mathbf{a}^2 = (-y_\xi, x_\xi), \quad (2)$$

and the volume

$$\tau = x_\xi y_\eta - x_\eta y_\xi, \quad (3)$$

where x_ξ, x_η, y_ξ and y_η are metrics of the coordinate transformation. Then the transformed equation of Eq. (1) takes the following form

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{f}}{\partial \xi} + \frac{\partial \tilde{g}}{\partial \eta} = \frac{\partial \tilde{Q}}{\partial t} + \sum_{l=1}^2 \frac{\partial \tilde{F}^l}{\partial \xi^l} = 0, \quad (4)$$

where

$$\tilde{Q} = \tau Q, \quad (\tilde{F}^1, \tilde{F}^2) = (\tilde{f}, \tilde{g}) = (\mathbf{a}^1 \cdot \mathbf{F}, \mathbf{a}^2 \cdot \mathbf{F}), \quad (\xi^1, \xi^2) = (\xi, \eta). \quad (5)$$

It is assumed that the physical domain Ω is partitioned into N non-overlapping elements Ω^i . Note that Eq. (1) holds on each Ω^i . When Ω^i is transformed into the corresponding standard element Ω_s , Eq. (4) holds on Ω_s . Also note that

the volume τ and metrics are constant on linear elements. Therefore, the approximation $Q \in \mathbb{Q}^k(\Omega_s)$ is equivalent to $Q \in \mathbb{Q}^k(\Omega^i)$ and $\tilde{Q} = \tau Q \in \mathbb{Q}^k(\Omega_s)$. Herein, $\mathbb{Q}^k(\Omega)$ is the space of tensor product of polynomials of degree at most k in each variable defined on Ω . Similar arguments work for f , g , $\tilde{f}$ and $\tilde{g}$ on linear elements. These relations will be used interchangeably depending on the contexts in the rest of this section.

Next we give a brief review of the DG, SD and CPR methods.

2.1. Quadrature-based discontinuous Galerkin (QDG) and nodal discontinuous Galerkin (NDG) methods

We approximate the exact solution of the conservation law using an element-wise continuous polynomial Q_h . Let W be an arbitrary weighting function or test function. The weighted residual form of the governing equations on each element Ω^i then reads

$$\int_{\Omega^i} \frac{\partial Q_h}{\partial t} W dV + \int_{\Omega^i} \nabla \cdot \mathbf{F}(Q_h) W dV = 0. \quad (6)$$

Applying integration by parts to Eq. (6), one obtains

$$\int_{\Omega^i} \frac{\partial Q_h}{\partial t} W dV - \int_{\Omega^i} \sum_{l=1}^2 \frac{\partial W}{\partial x^l} F^l(Q_h) dV + \int_{\partial\Omega^i} \mathbf{F} \cdot \mathbf{n} W dS = 0, \quad (7)$$

where $(x^1, x^2) = (x, y)$, $(F^1, F^2) = (f, g)$ and $\mathbf{n}$ is the outward unit normal vector of $\partial\Omega^i$.

It is clear that the surface integral in Eq. (7) is not properly defined as the numerical solution is discontinuous across element interfaces. In order to ensure conservation, the normal flux term $\mathbf{F} \cdot \mathbf{n}$ is replaced with a Riemann flux $F_{com}^n(Q_h^i, Q_h^{i+}, \mathbf{n})$, where Q_h^{i+} denotes the solution outside the current element Ω^i . Various (approximate) Riemann solvers can be used to calculate the Riemann flux, and the Roe Riemann solver [36] is adopted in this paper. Then Eq. (7) can be rewritten as

$$\int_{\Omega^i} \frac{\partial Q_h}{\partial t} W dV - \int_{\Omega^i} \sum_{l=1}^2 \frac{\partial W}{\partial x^l} F^l(Q_h) dV + \int_{\partial\Omega^i} F_{com}^n(Q_h^i, Q_h^{i+}, \mathbf{n}) W dS = 0. \quad (8)$$

In the DG approach, a finite-dimensional basis set $\{W_j\}$ is chosen as the solution space. Then the governing equation is projected onto each member of the basis set. Thus, Eq. (8) is reformulated as

$$\frac{\partial}{\partial t} \int_{\Omega^i} W_k \sum_j (Q_{h,j} W_j) dV - \int_{\Omega^i} \sum_{l=1}^2 \frac{\partial W_k}{\partial x^l} F^l(Q_h) dV + \int_{\partial\Omega^i} W_k F_{com}^n dS = 0. \quad (9)$$

The first integral in Eq. (9) is usually written as a multiplication of the mass matrix M and the time derivative of the solution vector $[Q_h]$. Herein, the square bracket $[\cdot]$ denotes the vector form corresponding to certain space discretization. The mass matrix M is of the form

$$M = [m_{kj}], \quad m_{kj} = \int_{\Omega^i} W_k W_j dV. \quad (10)$$

If $\mathbf{F}$ is a linear function of Q , then $\mathbf{F}$ can be expressed as $\mathbf{F} = \sum_j \mathbf{F}_j W_j$. Under this constraint, the second integral in Eq. (9) can be formulated as a multiplication of the stiffness matrix S^l and the flux vector F^l . The stiffness matrix S^l is of the form

$$S^l = [s_{kj}^l], \quad s_{kj}^l = \int_{\Omega^i} W_j \frac{\partial W_k}{\partial x^l} dV, \quad l = 1, 2. \quad (11)$$

However, if $\mathbf{F}$ is a nonlinear function of Q , then $\mathbf{F}$ cannot generally be expressed via the basis set $\{W_j\}$. Quadratures are used to compute the volume and surface integrals. Clearly these operations can be expensive, and some cost-effective approaches are required to improve the computational efficiency.

One such solution is the quadrature-free approach proposed in [14]. We assume that the solution space is $\mathbb{Q}^k(\Omega^i)$. As mentioned earlier, the nonlinear flux $\mathbf{F}$ cannot generally be expressed as polynomials. Here we assume that the flux $\mathbf{F}$ is a polynomial which belongs to the same space $\mathbb{Q}^k(\Omega^i)$ as that of the solution Q_h , and denote it by $\mathbf{F}_h$. Then Eq. (9) still holds for $\mathbf{F}_h$. We also assume that F_{com}^n belongs to the polynomial space $P^k(\partial\Omega^i)$ and can be expressed by the basis set $\{W_{f,j}\}$ as $F_{com,f}^n = \sum_j F_{com,f,j}^n W_{f,j}$ on each surface. Thus mass matrices B_f for the surface integration in Eq. (9) are formed as

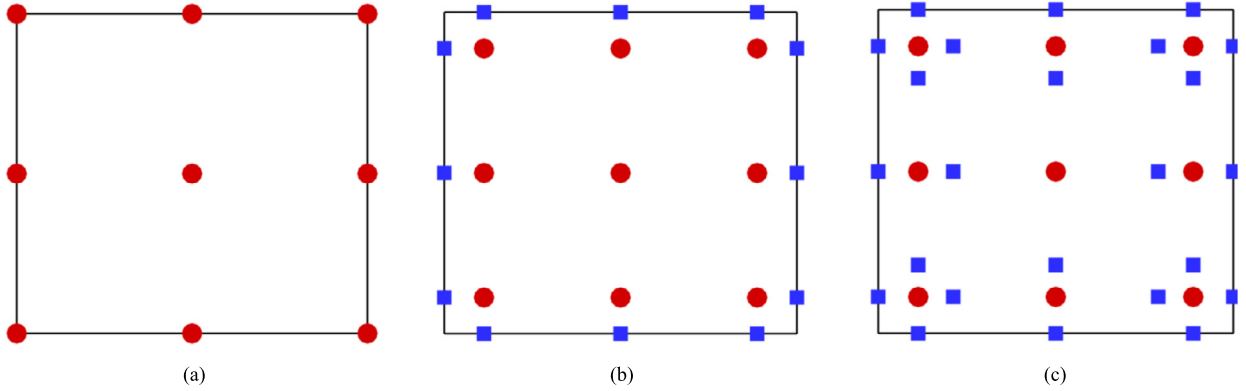


Fig. 1. Three types of collocation of the solution points (red dots) and interface flux points (blue squares) on quadrilateral elements. (a) Solution points: Lobatto points; Interface flux points (coincided with solution points): Lobatto points; (b) Solution points: Gauss points; Interface flux points: Gauss points; (c) Solution points: Gauss points; Flux points: Gauss points. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$B_f = [b_{f,kj}], \quad b_{f,kj} = \int_f W_k W_{f,j} dS_f. \quad (12)$$

Finally, Eq. (9) takes the following vector form

$$\frac{\partial [Q_h]}{\partial t} = M^{-1} \left(\sum_{l=1}^2 S^l [F^l] - \sum_f B_f [F_{com,f}^n] \right). \quad (13)$$

Note that in order to reduce the nonlinear aliasing errors, $\mathbf{F}$ and F_{com}^n can be approximated by polynomials of higher degree.

Nodal types of DOFs are used in the implementation of QDG. In view of the accuracy of the Gauss quadrature, a Gauss-point-based nodal type of DOFs is adopted here. Correspondingly, a set of Gauss-point-based Lagrange polynomials are chosen as the basis of the solution space. In addition, a Lobatto-point-based nodal type of DOFs is also tested here. In this approach, for a P^k reconstruction, the $k+1$ point Lobatto quadrature is used in each natural coordinate direction. Herein, P^k indicates that the solution belongs to the space of polynomials of degree at most k in each variable defined on the elements. This notation will be used in the rest of the paper without further explanation. Obviously, this quadrature rule is only accurate for the polynomial of degree at most $2k-1$ and is not exact for the integrals in Eq. (9). This scheme is different from NDG developed by Hesthaven and Warburton [2]. According to Ref. [2], NDG also chooses the Lobatto-point-based nodal type of DOFs. But in NDG, all matrices originated from Eq. (9) are calculated exactly. The allocation of Lobatto-point-based DOFs and that of Gauss-point-based DOFs in a quadrilateral cell are displayed in Fig. 1(a)–(b).

2.2. Spectral difference (SD) method

The SD scheme handles the conservation law via the finite-difference approach, which solves Eq. (4) on a set of solution points. In this scheme, the flux $\tilde{\mathbf{F}}$ is approximated by a polynomial $\tilde{\mathbf{F}}_h^I$ on the standard element Ω_s . Then Eq. (4) is reformulated as

$$\frac{\partial \tilde{Q}_h}{\partial t} + \sum_{l=1}^2 \frac{\partial \tilde{F}_h^{l,l}}{\partial \xi^l} = 0. \quad (14)$$

Two sets of points, namely solution points and flux points are used in the SD method. In this study, the solution points are chosen as Gauss points and the flux points are chosen as Gauss points plus two end points in a 1D element. The allocation of solution and flux points for 2D quadrilateral elements is shown in Fig. 1(c). The fundamental of the SD method is to approximate the solution with a degree k polynomial, which can be reconstructed from the $k+1$ solutions on the solution points, and to approximate the flux with a degree $k+1$ polynomial, which can be reconstructed from the $k+2$ fluxes on the flux points. Usually, both the solution and flux polynomials are formulated as the linear combination of Lagrange polynomials. The final form of the SD scheme for a standard quadrilateral cell at each solution point (i, j) is written as

$$\frac{\partial \tilde{Q}_{h,i,j}}{\partial t} + \sum_{m=1}^{k+2} \left[\tilde{f}_{h,m,j}^I \left(\frac{\partial}{\partial \xi} l_{FPs,m}(\xi) \right)_i + \tilde{g}_{h,i,m}^I \left(\frac{\partial}{\partial \eta} l_{FPs,m}(\eta) \right)_j \right] = 0, \quad (15)$$

where l_{FPs} stands for the flux points based Lagrange polynomial. Readers are referred to Refs. [20,37] for more details on the SD method.

2.3. Correction procedure via reconstruction (CPR) method

CPR can be derived independently from both the integral-based (Eq. (8)) and differential-based (Eq. (4)) forms of the conservation law. Here we only derive the formula via the weighted residual approach from Eq. (8). This approach will shed light on the theoretical foundation of the modification of CPR for the curved element as will be presented in the next section.

On using integration by part on the weak form as shown in Eq. (8), one obtains the strong form as follows

$$\int_{\Omega^i} \frac{\partial Q_h}{\partial t} W dV + \int_{\Omega^i} \nabla \cdot \mathbf{F}(Q_h) W dV + \int_{\partial\Omega^i} (F_{com}^n - F^n) W dS = 0, \quad (16)$$

where $F^n = \mathbf{F} \cdot \mathbf{n}$.

The next step is critical in the elimination of the test function. The surface integral in Eq. (16) is cast into the form of a volume integral via the introduction of a correction field, i.e., $\delta^C \in \mathbb{Q}^k(\Omega^i)$, on the element Ω^i . This is expressed as

$$\int_{\Omega^i} \delta^C W dV = \int_{\partial\Omega^i} (F_{com}^n - F^n) W dS. \quad (17)$$

Efficient ways for the reconstruction of δ^C can be found in [3,27].

On substituting Eq. (17) into Eq. (16), one obtains

$$\int_{\Omega^i} W \left(\frac{\partial Q_h}{\partial t} + \nabla \cdot \mathbf{F}(Q_h) + \delta^C \right) dV = 0. \quad (18)$$

If $\mathbf{F}$ is a linear function of Q , then $\nabla \cdot \mathbf{F} \in \mathbb{Q}^k(\Omega^i)$. In this case, the terms enclosed by the parentheses in Eq. (18) are all elements of $\mathbb{Q}^k(\Omega^i)$. For nonlinear conservation laws, as discussed in Section 2.1, $\nabla \cdot \mathbf{F}$ is generally not an element of $\mathbb{Q}^k(\Omega^i)$. In this case, the natural choice is to project $\nabla \cdot \mathbf{F}$ onto $\mathbb{Q}^k(\Omega^i)$. Denote $\mathbb{P}(\nabla \cdot \mathbf{F})$ a projection of $\nabla \cdot \mathbf{F}$ to $\mathbb{Q}^k(\Omega^i)$. One choice is

$$\int_{\Omega^i} \mathbb{P}(\nabla \cdot \mathbf{F}) W dV = \int_{\Omega^i} \nabla \cdot \mathbf{F} W dV. \quad (19)$$

Thus Eq. (18) reduces to

$$\frac{\partial Q_h}{\partial t} + \mathbb{P}(\nabla \cdot \mathbf{F}) + \delta^C = 0. \quad (20)$$

Two efficient approaches [26,27], namely the Lagrange polynomial (LP) approach and the chain-rule (CR) approach, can be used to calculate the projection of the flux divergence. In the LP approach, the flux function $\mathbf{F}$ is assumed to belong to the space $\mathbb{Q}^k(\Omega^i)$ and can be expanded with the linearly independent basis consisting of Lagrange polynomials. The projection of the flux divergence is then expressed as

$$\mathbb{P}(\nabla \cdot \mathbf{F}) = \nabla \cdot \left(\sum_{j \in SP} L_j \mathbf{F}_j \right), \quad (21)$$

where ‘SP’ stands for the set of solution points. Note that for a quadrilateral element, L_j is a tensor product of two 1D Lagrange polynomials, $l_j(\xi)$ and $l_j(\eta)$, associated with the solution point ‘j’. In the CR approach, the flux divergence $\nabla \cdot \mathbf{F}$ is assumed to belong to the space $\mathbb{Q}^k(\Omega^i)$ and is expressed with Lagrange polynomials as

$$\mathbb{P}(\nabla \cdot \mathbf{F}) = \sum_{j \in SP} L_j \left(\frac{\partial \mathbf{F}}{\partial Q} \nabla Q \right)_j. \quad (22)$$

Note that for the linear conservation law, the LP and CR approaches achieve the same results. According to Jameson et al. [38], the LP approach can deteriorate the accuracy for nonlinear fluxes due to aliasing errors. The CR approach can enhance the accuracy for CPR using Lobatto points, but makes the corresponding CPR scheme non-conservative [27]. A conservative remedy has been shown by Gao and Wang [39] recently.

Two sets of allocation of solution points and interface flux points are used, as shown in Fig. 1(a)–(b). It is clear that the scheme using Lobatto points can be more numerically efficient than the scheme using Gauss points, as in the former scheme, interface flux points coincide with the solution points, and no interpolation from the solution points to the flux points is needed when calculating the interface fluxes. However, for the nonlinear conservation law, CPR using Gauss point might gain certain accuracy benefits by minimizing the aliasing errors due to the allocation of solution points.

3. Analysis of several schemes on curved elements

As discussed in Section 1, proper treatment of the curved elements near wall boundaries is essential for a successful high-order method [10,11,40,41]. The key challenge brought in by the curved element is that the nonlinear geometry can degrade the accuracy of the solution and large numerical errors might be generated if the geometric metrics are not fully resolved. The situation can become even worse as the approximation degree of the curved element becomes higher. A natural thought to overcome such difficulties is to increase the approximation degree of the solution. However, it is not economic and even not practical to compensate the errors from nonlinear geometry by increasing the approximation degree of the solution, as the computational cost will grow quadratically/cubically with the polynomial degree for a quadrilateral/hexahedral element. Therefore, it is desirable to design a scheme that can both achieve reasonable accuracy and minimize the additional computational cost on the curved element when compared with the corresponding scheme on the linear element.

Before moving forward, we examine and clarify the problems encountered in the flux reconstruction process on curved elements. First of all, one might realize the fact that $\tilde{Q} = \tau Q$ can be approximated by the space $\mathbb{Q}^k(\Omega_s)$, and Q can be approximated both by the spaces $\mathbb{Q}^k(\Omega^i)$ and $\mathbb{Q}^k(\Omega_s)$. In consideration of efficiency, $\mathbb{Q}^k(\Omega_s)$ is used to approximate $\tilde{Q}$ or Q . By using the identity

$$\nabla \cdot \mathbf{F} = \frac{1}{\tau} \sum_{l=1}^2 \mathbf{a}^l \cdot \frac{\partial \mathbf{F}}{\partial \xi^l}, \quad (23)$$

the conservation law in physical and computational domain can be transformed from each other mathematically. However, for actual numerical approximation on the curved element, these two approaches, namely assuming that $\tilde{Q} \in \mathbb{Q}^k(\Omega_s)$ and assuming that $Q \in \mathbb{Q}^k(\Omega_s)$, can be quite different. This is attributed to the fact that τ for the curved element is actually a polynomial. If one assumes that $\tilde{Q} \in \mathbb{Q}^k(\Omega_s)$, then τ can significantly degrade the accuracy of Q . This will be demonstrated shortly by the results from one implementation of the SD method, which follows the assumption that $\tilde{Q}$ is a polynomial sitting in the space $\mathbb{Q}^k(\Omega_s)$.

The immediate question arising from the assumption $Q \in \mathbb{Q}^k(\Omega_s)$ is how to ensure that the flux reconstruction is accurate. In order to answer this question, we first investigate the properties of QDG.

3.1. QDG on curved elements

On using Eq. (5), one can rewrite Eq. (9) on a standard element Ω_s in the following form,

$$\frac{\partial}{\partial t} \int_{\Omega_s} W_k \sum_j (Q_{h,j} W_j) \tau dV_s - \int_{\Omega_s} \sum_{l=1}^2 \frac{\partial W_k}{\partial \xi^l} (\mathbf{a}^l \cdot \mathbf{F}(Q_h)) dV_s + \int_{\partial \Omega_s} W_k \tilde{F}_{num,f} dS_s = 0. \quad (24)$$

Herein, $\tilde{F}_{num,f}$ is the numerical flux function on the surface of the standard element and is defined as

$$\tilde{F}_{num,f} = F_{com}^n | \mathbf{a}^f |. \quad (25)$$

A common practice to handle Eq. (24) on the curved element is to conduct the volume and surface integrals using the quadrature rule with sufficient accuracy. As a matter of fact, the computational cost can be unacceptably high. Therefore, it is desirable that the mass, stiffness and surface mass matrices are explicitly formed as discussed in Section 2.1. The formation of the mass and stiffness matrices is straightforward. The mass matrix $\tilde{M}$ takes the following form,

$$\tilde{M} = [\tilde{m}_{kj}], \quad \tilde{m}_{kj} = \int_{\Omega_s} W_k W_j \tau dV_s. \quad (26)$$

If $\mathbf{F}(Q_h)$ is assumed to belong to the same space as that of the solution Q_h , then it can be expressed as $\mathbf{F}_h = \sum_j \mathbf{F}_{h,j} W_j$. On using the definition of the area vector $\mathbf{a}$, we obtain the stiffness matrix $\tilde{S}^l$ as follows

$$\tilde{S}^l = [\tilde{s}_{kj}^l], \quad \tilde{s}_{kj}^l = \int_{\Omega_s} W_j \frac{\partial W_k}{\partial \xi^l} \tau dV_s, \quad l = 1, 2. \quad (27)$$

Now our main attention will focus on the surface integration of Eq. (24). On curved elements, both F_{com}^n and $| \mathbf{a}^f |$ are generally not polynomials. Therefore, a simple way to handle the surface integration is to use the quadrature rules with more quadrature points. But it is clear that if the quadrature rule is adopted, the mass surface matrix $\tilde{B}_f$ will not be explicitly formed and then $\tilde{M}^{-1}$ cannot be pre-multiplied with $\tilde{B}_f$ to save computational cost. In order to improve the efficiency of QDG, the mass surface matrix $\tilde{B}_f$ is desirable to be formed explicitly. We note that if $\mathbf{F} = (f, g)$ are assumed to be polynomials belonging to $\mathbb{Q}^k(\Omega_s)$, then $\tilde{F}_{num,f}$ is also a polynomial, and its degree depends both on the degree of the flux polynomial and the degree of the curved element, which can be obtained from Table 1. Note that $P^{k/e}(\xi)$ or $P^{k/e}(\eta)$ in

Table 1

Approximation degree of the flow related and geometry related variables.

	Solution Q	Fluxes f, g	Geometry x, y	Jacobian $ J $	Metrics	
					x_ξ, y_ξ	x_η, y_η
Polynomial degree	$P^k(\xi) \times P^k(\eta)$	$P^k(\xi) \times P^k(\eta)$	$P^e(\xi) \times P^e(\eta)$	$P^{2e-1}(\xi) \times P^{2e-1}(\eta)$	$P^{e-1}(\xi) \times P^e(\eta)$	$P^e(\xi) \times P^{e-1}(\eta)$

Table 1 is the space of polynomials of degree at most k/e in ξ or η , respectively. In the implementation, F_{com}^n and $|\mathbf{a}^f|$ are computed separately on quadrature points, for which the quadrature rule is sufficiently accurate for the polynomial $\tilde{F}_{num,f}$. In view of the procedure as shown in Section 2.1, the mass surface matrix $\tilde{B}_f$ can be explicitly formed as below

$$\tilde{B}_f = [\tilde{b}_{f,kj}], \quad \tilde{b}_{f,kj} = \int_{f_s} W_k \tilde{W}_{f,j} dS_{f_s}. \quad (28)$$

Note that here the basis set $\{\tilde{W}_{f,j}\}$ should be chosen from the polynomial space approximating $\tilde{F}_{num,f}$. Similarly as for Eq. (13), the vector form of Eq. (24) reads

$$\frac{\partial[Q_h]}{\partial t} = \tilde{M}^{-1} \left(\sum_{l=1}^2 \tilde{S}^l[F^l] - \sum_f \tilde{B}_f[\tilde{F}_{num,f}] \right). \quad (29)$$

Thus, a quadrature-free approach for the curved element is formed for QDG, as all matrices can be calculated in the preprocessing stage.

3.2. CPR on curved elements

Based on the analyses of QDG, a modified CPR scheme for curved elements can be derived following the weighted residual method as follows. The key step in deriving the CPR scheme is to lift the surface corrections onto the interior solution points, as shown in Eq. (17). On rewriting this formula by assuming that $\delta^C \in \mathbb{Q}^k(\Omega_s)$, one obtains

$$\int_{\Omega_s} \delta^C W \tau dV_s = \int_{\partial\Omega_s} (F_{com}^n - F^n) |\mathbf{a}^f| W dS_s. \quad (30)$$

As previously mentioned, for the curved element the surface integration is commonly calculated via the quadrature rule. Thus Eq. (30) must be solved for each iteration, as no pre-stored correction coefficients can be used. This will inevitably deteriorate the performance of the CPR scheme. If one takes advantage of the fact that $(F_{com}^n - F^n) |\mathbf{a}^f|$ is a polynomial, the difficulty from the surface integration can be avoided. Then the mass matrix for the surface integration can be formulated from Eq. (28).

Remark 3.1. The developed CPR scheme for the curved element is a mixed scheme of the original CPR scheme and QDG scheme. The calculation of the flux divergence term follows the tensor-product-based (line-based) approach from the CPR scheme while the correction field is directly calculated from Eq. (30) as in the QDG scheme. The aim in designing this mixed approach is to enhance the numerical accuracy of the tensor-product-based scheme on curved elements while not significantly increasing the computational cost.

4. Implementation procedures for DG, SD and CPR methods

In this section, we show the implementation procedures for the QDG, SD and CPR methods. We also clarify the assumptions and some important formulas in different implementations of these methods.

Major operations in QDG are depicted as follows:

- Evaluation of the volume integration as indicated by the second term in Eq. (9) or Eq. (24). This term can always be formulated as a matrix–vector product.
- Evaluation of the surface integration as indicated by the third term in Eq. (9) or Eq. (24). Two major operations are conducted in this step. First of all, Riemann fluxes are evaluated on all surfaces. Then the surface integration can be formulated as a matrix–vector product or by the numerical quadrature rule.
- Update of the unknown DOFs. If the quadrature-free approach is carried out, then the inverse of the mass matrix as shown in Eq. (10) or Eq. (26) can be pre-multiplied to the stiffness matrix and to the mass matrix for the surface integration in the preprocessing stage. In this approach, a cheap point-wise update is needed. If the general quadrature rule based approach is adopted, a matrix–vector product is needed to update DOFs.

In the implementation of the QDG method, we always assume that $Q \in \mathbb{Q}^k(\Omega_s)$. In the quadrature-free based approach, all matrices are strictly formed based on the analyses in Section 2 and Section 3. In the quadrature rule based approach, sufficient quadrature points are used in volume and surface integrals to ensure accuracy.

Major operations in SD are stated as follows:

- Evaluation of the Riemann fluxes on all cell surfaces.
- Evaluation of the flux divergences as indicated by the second term in Eq. (15). Two major operations are carried out in this step. First, the conservative variables need to be interpolated from the solution points to the flux points. Then the flux divergences on the solution points are computed by differentiating the flux polynomial constructed from the fluxes on the flux points. Note that in Refs. [22,23] it is shown that the linear stability of the SD scheme is independent of the location of the solutions points. Therefore, the conservative variable interpolation can be omitted by collocating the solution and flux points. Here only the staggered-grid based SD scheme is adopted.
- Update of the unknown DOFs via a point-wise procedure.

We note that the reconstruction of the fluxes from the second step can result in different implementation of the SD method on curved elements. Specifically, if we assume that $\tilde{Q} \in \mathbb{Q}^k(\Omega_s)$, then $\tilde{Q}$ on the flux points are directly interpolated from those on solution points. This implementation procedure is named as ‘SD1’. If we assume that $Q \in \mathbb{Q}^k(\Omega_s)$, then Q are interpolated from the solution points to the flux points for the flux reconstruction. In this approach, $\partial \tilde{Q} / \partial t$ in Eq. (15) is reformulated as $\tau \partial Q / \partial t$ and it is termed as ‘SD2’. In both approaches, fluxes $\tilde{f}_h^I$ and $\tilde{g}_h^I$ in Eq. (15) are assumed to be polynomials as shown in Ref. [37]. The schemes SD1 and SD2 are designed to distinguish the performance difference of various approximation assumptions.

Major operations in CPR are summarized as follows:

- Evaluation of the Riemann fluxes on all cell surfaces.
- Evaluation of the flux divergences as indicated by the second term in Eq. (20). In the LP approach, the flux divergences are computed by differencing the flux polynomial constructed by the fluxes computed from the solution points. In the CR approach, the flux divergences are calculated analytically.
- Evaluation of the correction field as indicated by the third term in Eq. (20) via a point-wise procedure.
- Update of the unknown DOFs via a point-wise procedure.

In the implementation of the CPR method, we assume that $Q \in \mathbb{Q}^k(\Omega_s)$. In the calculation of $\mathbb{P}(\nabla \cdot \mathbf{F})$, it is assumed that $\mathbf{F} \in \mathbb{Q}^k(\Omega_s)$ in the LP approach and $\nabla \cdot \mathbf{F} \in \mathbb{Q}^k(\Omega_s)$ in the CR approach. Two types of approximation of δ^C are considered here. In the first approach, we assume that $\tau \delta^C \in \mathbb{Q}^k(\Omega_s)$. Thus δ^C on a quadrilateral element is specified as

$$\delta^C = \frac{1}{\tau} \left(\frac{\partial \tilde{f}^C}{\partial \xi} + \frac{\partial \tilde{g}^C}{\partial \eta} \right), \quad (31)$$

with

$$\begin{aligned} \tilde{f}^C(\xi, \eta) &= (\tilde{f}_{num,L}(\eta) - \tilde{f}(-1, \eta))G_L(\xi) + (\tilde{f}_{num,R}(\eta) - \tilde{f}(1, \eta))G_R(\xi), \\ \tilde{g}^C(\xi, \eta) &= (\tilde{g}_{num,L}(\xi) - \tilde{g}(\xi, -1))G_L(\eta) + (\tilde{g}_{num,R}(\xi) - \tilde{g}(\xi, 1))G_R(\eta). \end{aligned} \quad (32)$$

Herein, $\tilde{f}_{num,L}$, $\tilde{f}_{num,R}$, $\tilde{g}_{num,L}$ and $\tilde{g}_{num,R}$ are numerical flux functions evaluated on different element surfaces. G_L and G_R are named as correction functions, which belong to the polynomial space of degree $(k+1)$. In the present implementation, G_L and G_R are chosen as right and left Radau polynomials. It is clear that δ^C is reconstructed in the one-dimensional sense. This approach is referred as the original CPR formulation, and indicated by ‘CPR...Ori’. Readers are referred to Refs. [3,22] for more details on this approach.

In the second approach, we assume that $\delta^C \in \mathbb{Q}^k(\Omega_s)$. For linear elements, τ and $\mathbf{a}$ are constants in one element and Eq. (31) can still be used to evaluate δ^C without losing any accuracy. In fact, it has been proved in Ref. [31] that Eq. (31) is equivalent to Eq. (17) or Eq. (30). However, for curved elements, implementations using Eq. (30) or Eq. (31) are different. As discussed in Section 3.2, if Eq. (30) is used, δ^C is evaluated using the following form,

$$[\delta^C] = \tilde{M}^{-1} \sum_f \tilde{B}_f [\Delta \tilde{F}_f], \quad (33)$$

where $\Delta \tilde{F}_f = (F_{com}^n - F^n) |\mathbf{a}^f| \text{sign}(\mathbf{a}^f \cdot \mathbf{n})$. This approach is generally not one-dimensional for curved elements. As will be shown shortly, this implementation can enhance the accuracy of the LP based CPR scheme on highly skewed elements. This approach is referred as the modified CPR formulation, and denoted by ‘CPR...Mod’.

For clarity, we summarize all the schemes which will be compared in the next section. These schemes are generally divided into three categories, namely DG, SD and CPR. Further subdivisions of these methods on linear and curved elements are listed as follows.

Schemes from the DG family on linear elements include DG using Gauss quadrature (denoted as QDG), DG using Lobatto quadrature (denoted as QDG_LB) and NDG. The main features of these schemes are illustrated as below:

	QDG	NDG	QDG_LB
Solution points	Gauss	Lobatto	Lobatto
Solution space	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$
Flux space	$f, g \in \mathbb{Q}^k(\Omega_s)$	$f, g \in \mathbb{Q}^k(\Omega_s)$	$f, g \in \mathbb{Q}^k(\Omega_s)$
Exact stiffness matrix	Yes	Yes	No
Exact surface mass matrix	Yes	Yes	No
Quadrature-free implementation	Yes	Yes	No

As discussed previously, the flux reconstruction procedures in QDG can be flexible, and different reconstruction methods can result in different numerical cost and accuracy. Three types of QDG are compared on curved elements. Two of them assume that the fluxes are approximated by polynomials $\mathbf{F}_h$ which sit in the same space as that of the solution Q_h , with one explicitly forming the surface mass matrix in the preprocessing stage (i.e. the quadrature-free approach, denoted as QDG) and the other carrying out the surface integration via the quadrature rule (denoted as QDG_Q). The third type of QDG (denoted as QDG_H) assumes that the degree of the flux polynomial $\mathbf{F}_h$ is one order higher than that of the solution Q_h and the quadrature-free approach is adopted to update DOFs. This scheme is designed to examine the effects of quadrature rules on aliasing errors control for the nonlinear Euler simulation. The main features of these schemes are summarized as below:

	QDG	QDG_Q	QDG_H
Solution points	Gauss	Gauss	Gauss
Solution space	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$
Flux space	$f, g \in \mathbb{Q}^k(\Omega_s)$	$f, g \in \mathbb{Q}^k(\Omega_s)$	$f, g \in \mathbb{Q}^{k+1}(\Omega_s)$
Exact stiffness matrix	Yes	Yes	Yes
Exact surface mass matrix	Yes	Not explicitly formed, but sufficient quadrature rule is used	Yes
Quadrature-free implementation	Yes	No	Yes

Schemes from the SD family on both linear and nonlinear elements include SD1 and SD2. The main features of these schemes are illustrated as below:

	SD1	SD2
Solution points	Gauss	Gauss
Flux points	Gauss & two end points	Gauss & two end points
Solution space	$\bar{Q} \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$
Flux space	$\bar{f}, \bar{g} \in \mathbb{Q}^{k+1}(\Omega_s)$	$\bar{f}, \bar{g} \in \mathbb{Q}^{k+1}(\Omega_s)$

Schemes from the CPR family on linear elements include LP and CR based CPR using Gauss points (denoted as CPR_GS_LP and CPR_GS_CR respectively) and LP and CR based CPR using Lobatto points (denoted as CPR_LB_LP and CPR_LB_CR respectively). The main features of these schemes are illustrated as below:

	CPR_GS_LP	CPR_GS_CR	CPR_LB_LP	CPR_LB_CR
Solution points	Gauss	Gauss	Lobatto	Lobatto
Flux divergence reconstruction	LP	CR	LP	CR
Solution space	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$
Local flux space	$f, g \in \mathbb{Q}^k(\Omega_s)$	$\nabla \cdot \mathbf{F} \in \mathbb{Q}^k(\Omega_s)$	$f, g \in \mathbb{Q}^k(\Omega_s)$	$\nabla \cdot \mathbf{F} \in \mathbb{Q}^k(\Omega_s)$
Correction field space	$\tilde{\delta}_c \in \mathbb{Q}^k(\Omega_s)$	$\tilde{\delta}_c \in \mathbb{Q}^k(\Omega_s)$	$\tilde{\delta}_c \in \mathbb{Q}^k(\Omega_s)$	$\tilde{\delta}_c \in \mathbb{Q}^k(\Omega_s)$

Schemes from the CPR family on curved elements include CPR_GS_LP_Ori, CPR_GS_CR_Ori, CPR_LB_LP_Mod and CPR_LB_CR_Mod. Note that the schemes CPR_GS_LP_Ori and CPR_GS_CR_Ori are equivalent to CPR_GS_LP and CPR_GS_CR on linear elements, respectively. ‘_Ori’ is merely used to distinguish these schemes with the corresponding modified CPR schemes (denoted by ‘_Mod’) proposed in Section 3.2. The main features of these schemes are illustrated as below:

	CPR_GS_LP_Ori	CPR_GS_CR_Ori	CPR_GS_LP_Mod	CPR_GS_CR_Mod
Solution points	Gauss	Gauss	Gauss	Gauss
Flux divergence reconstruction	LP	CR	LP	CR
Solution space	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$	$Q \in \mathbb{Q}^k(\Omega_s)$
Local flux space	$f, g \in \mathbb{Q}^k(\Omega_s)$	$\nabla \cdot \mathbf{F} \in \mathbb{Q}^k(\Omega_s)$	$f, g \in \mathbb{Q}^k(\Omega_s)$	$\nabla \cdot \mathbf{F} \in \mathbb{Q}^k(\Omega_s)$
Correction field space	$\tilde{\delta}_c \in \mathbb{Q}^k(\Omega_s)$	$\tilde{\delta}_c \in \mathbb{Q}^k(\Omega_s)$	$\delta_c \in \mathbb{Q}^k(\Omega_s)$	$\delta_c \in \mathbb{Q}^k(\Omega_s)$

In order to minimize the influence of detailed implementation on the relative performance of different schemes, the same implementation procedures are ensured for the same operations encountered in different schemes, e.g., interpolation,

differentiation and matrix–vector multiplication, etc. Besides, the optimization option ‘-O3’ is turned on for all simulations while compiling using ‘g++’.

5. Numerical results and discussions

Accuracy and efficiency analyses on several high-order methods, namely DG, weighted essentially nonoscillatory (WENO) schemes [42], SV and SD, have been carried out in previous works [43–46]. But to the best of the authors’ knowledge, no comprehensive comparisons on numerical performance of DG, SD and CPR are yet available. In the present paper, estimations of the algorithm accuracy and efficiency based on numerical tests are carried out as given below.

5.1. Problem statements

Both linear and nonlinear hyperbolic conservation laws are studied on different grids for accuracy and efficiency evaluation. The first test is carried out on the wave propagation problem with constant wave speed in the form of Eq. (1) with

$$Q = u, \quad f = c_x u, \quad g = c_y u, \quad (34)$$

where c_x and c_y are constant wave speeds in x and y directions respectively. The exact solution is specified as

$$u(\mathbf{x}, t) = \sin \left[2\pi t - \pi \left(\frac{x}{c_x} + \frac{y}{c_y} \right) \right]. \quad (35)$$

The problem evolves from $t = 0$ and the time varying analytical solutions are enforced on boundaries. The computation domain is set to be $[-5, 5] \times [-5, 5]$.

The second test is performed for the nonlinear hyperbolic conservation law. Consider the 2D Euler equations in the form of Eq. (1) with

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{bmatrix}, \quad g = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{bmatrix}, \quad (36)$$

where ρ is the density, u and v are the velocity components in x and y directions, respectively, p is the pressure and E is the total energy. The pressure is related to the total energy by the perfect gas law,

$$E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho (u^2 + v^2), \quad (37)$$

with the ratio of specific heats $\gamma = 1.4$.

A 2D Euler vortex propagation case is adopted to test the accuracy and numerical efficiency of the discontinuous high-order methods. In view of Ref. [47], the definition of the isentropic vortex and its evolution are described as follows. The mean flow is specified with $\{\rho, u, v, p\} = \{1, 1, 1, 1\}$. The analytical solution of the perturbations on the velocity (u, v), the temperature $T = p/\rho$, and the entropy $S = p/\rho^\gamma$ are specified as

$$(\delta u, \delta v) = \frac{\varepsilon}{2\pi} e^{0.5(1-r^2)} (-y_r, x_r), \quad \delta T = -\frac{(\gamma - 1)\varepsilon^2}{8\gamma\pi^2} e^{1-r^2}, \quad \delta S = 0. \quad (38)$$

Herein, $r^2 = x_r^2 + y_r^2$ with $x_r = x - ut$ and $y_r = y - vt$. The vortex strength $\varepsilon = 5$. The computation domain is set to be $[-5, 5] \times [-5, 5]$. The problem evolves from $t = 0$ and time varying analytical solutions are enforced on the boundaries.

An explicit multi-stage strong stability preserving (SSP) Runge–Kutta scheme [48] is employed for time integration. The three-stage Runge–Kutta scheme is expressed as follows

$$\begin{aligned} Q^{(1)} &= Q^n + \Delta t R(Q^n), \\ Q^{(2)} &= \frac{3}{4} Q^n + \frac{1}{4} [Q^{(1)} + \Delta t R(Q^{(1)})], \\ Q^{n+1} &= \frac{1}{3} Q^n + \frac{2}{3} [Q^{(2)} + \Delta t R(Q^{(2)})]. \end{aligned} \quad (39)$$

The L_2 error is used to evaluate the accuracy of all schemes in the paper. For any solution variable s , the L_2 error is defined in the integral sense as

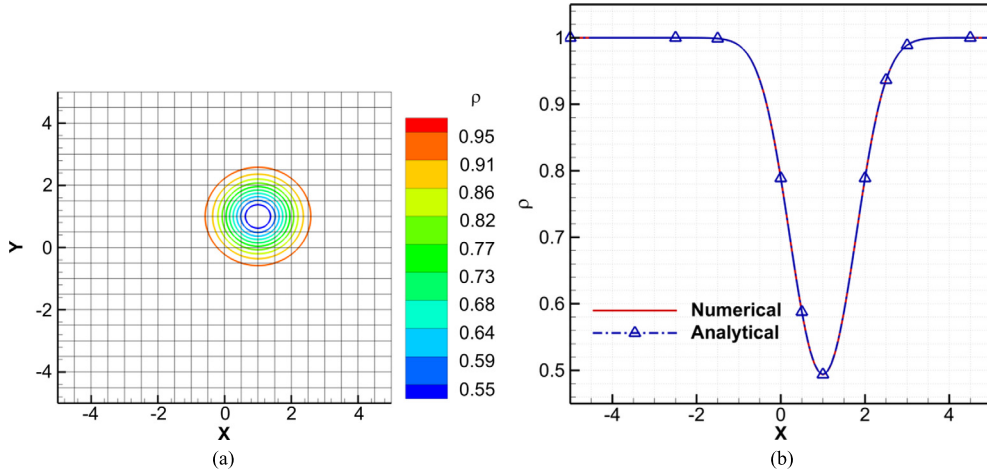


Fig. 2. Density fields at $t = 1.0$ from both numerical and analytical results. (a) The numerical solution with P^4 CPR_GS_LP reconstruction on a 20×20 mesh. (b) Comparison between the analytical solution and the numerical solution at $y = 2.0$.

$$Error_{L_2(\Omega)} = \sqrt{\frac{\int_{\Omega} (s - s^{exact})^2 dV}{\int_{\Omega} dV}}. \quad (40)$$

The work unit is adopted to measure the computational cost of the schemes. As suggested by the 1st International Workshop on High-Order CFD Methods [35], the work unit is defined by

$$work\ unit = CPU\ time / TauBench\ time.$$

Herein, ‘CPU time’ measures the wall clock time taken by the CFD solver (excluding the initialization, post-processing data preparation time and file I/O time) and ‘TauBench time’ is the wall clock time averaged from ten executions of the unstructured grid benchmark TauBench [49]. For the TauBench case, the parameters are set up as follows:

$$./TauBench -n 250000 -s 10$$

5.2. Accuracy and efficiency evaluation on linear elements

Numerical results of several schemes from CPR, DG and SD families are compared on meshes tessellated with linear quadrilateral elements. These schemes include CPR_GS_LP, CPR_GS_CR, CPR_LB_LP and CPR_LB_CR from the CPR family; QDG, QDG_LB and NDG from the DG family; SD1 and SD2 from the SD family. Descriptions of these schemes have been detailed in Section 4.

It is found that on linear quadrilateral elements, whether with constant or linear metrics, the conclusions drawn from both linear and nonlinear conservation laws are similar. Therefore, only results from the Euler vortex propagation problem are displayed in this section for conciseness.

5.2.1. Accuracy and efficiency evaluation on regular linear elements

The numerical simulations are carried out until $t = 1.0$. A sufficiently small time step $\Delta t = 4 \times 10^{-4}$ is chosen to ensure that the space discretization error dominates during the simulations. The density contour from the numerical solution on a 20×20 mesh with the P^4 CPR_GS_LP scheme at $t = 1.0$ is displayed in Fig. 2(a). The comparison of density fields from both the analytical and numerical results at $t = 1.0$ and $y = 1.0$ is shown in Fig. 2(b).

- Note that if the solution points in the CPR family coincide with the quadrature points in the DG family, then the scheme CPR_GS_LP is equivalent to the scheme QDG and the scheme CPR_LB_LP is equivalent to NDG on linear elements. This is numerically examined for both the linear and nonlinear conservation laws on meshes tessellated with regular quadrilateral elements. The results on a 20×20 mesh with P^4 solution reconstruction are presented in Table 2. From this test, we conclude that on linear quadrilateral elements, the schemes from DG family are essentially one-dimensional in each natural coordinate direction. In fact, this conclusion can also be drawn by examining the matrix types of QDG.
- From Table 2 it is also found that the CR approach can enhance the accuracy of CPR using Lobatto points, and can save computational cost when compared with the LP approach.

Table 2

Comparison of the schemes from CPR and DG families on a 20×20 mesh tessellated with linear elements of constant Jacobian. P^4 reconstruction is adopted and all cases are evolved till $t = 1.0$.

		CPR_GS_CR	CPR_GS_LP	QDG
Linear advection	L_2 Error (u)	$3.31921448975 \times 10^{-8}$	$3.31921448975 \times 10^{-8}$	$3.31921447686 \times 10^{-8}$
	Work unit	0.58	0.58	1.44
Euler	L_2 Error (ρ)	$1.1415877664 \times 10^{-6}$	$1.13867865928 \times 10^{-6}$	$1.13867865929 \times 10^{-6}$
	Work unit	2.71	3.70	7.31
		CPR_LB_CR	CPR_LB_LP	NDG
Linear advection	L_2 Error (u)	$3.3192144851 \times 10^{-8}$	$3.3192144851 \times 10^{-8}$	$3.31921449137 \times 10^{-8}$
	Work unit	0.48	0.48	1.21
Euler	L_2 Error (ρ)	$1.70485672474 \times 10^{-6}$	$6.63911845006 \times 10^{-6}$	$6.63911845009 \times 10^{-6}$
	Work unit	2.08	2.44	6.66

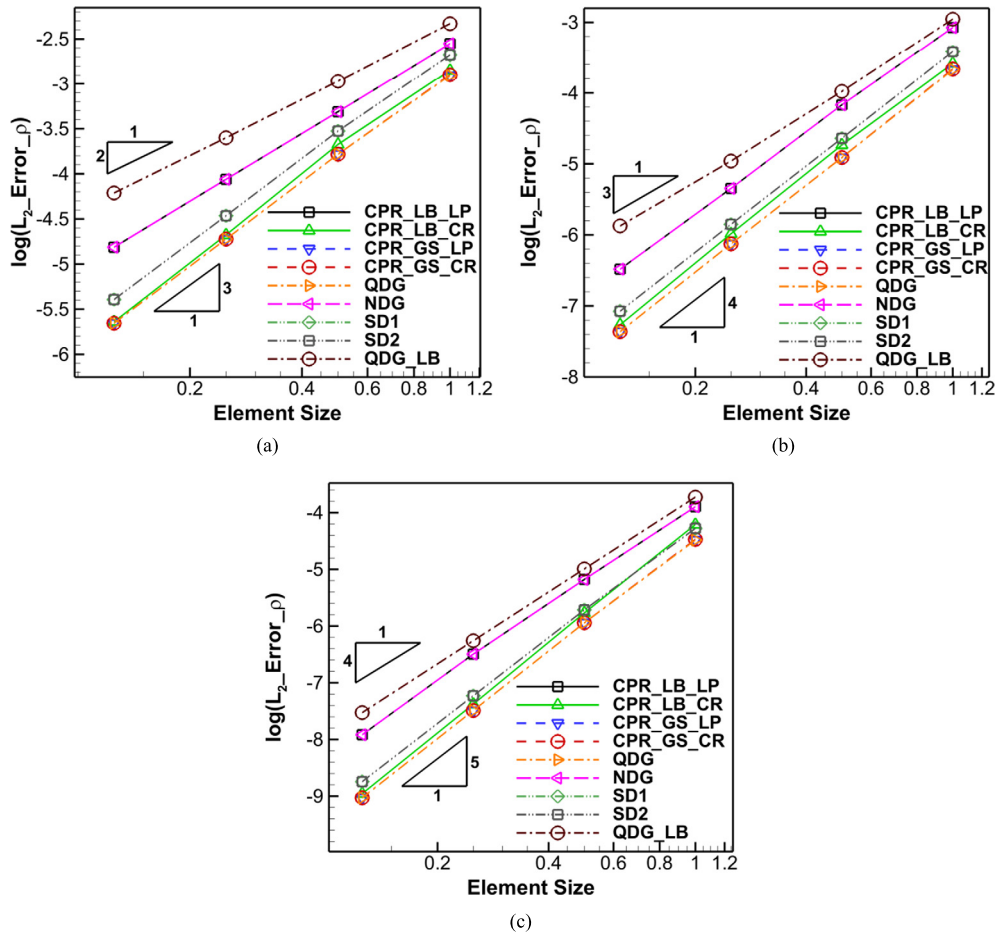


Fig. 3. Convergence test of the Euler vortex propagation problem for (a) P^2 , (b) P^3 and (c) P^4 reconstructions on meshes tessellated with regular linear elements.

The convergence rates of P^2 to P^4 reconstructions for the aforementioned schemes are shown in Fig. 3. From this figure, it is found that the convergence rate for the scheme QDG_LB is one order lower than the optimal one due to the insufficient integral accuracy. The convergence rates of the schemes CPR_LB_LP and NDG are about half an order lower than the optimal one. All other schemes achieve optimal convergence rate $k + 1$ for a P^k solution reconstruction. The schemes QDG, CPR_LB_CR and the schemes from the CPR family with Gauss points, i.e. CPR_GS_LP and CPR_GS_CR achieve smaller absolute errors than other schemes.

Fig. 4 shows the relation between the absolute error and the work unit for all schemes. Several conclusions can be drawn as follows:

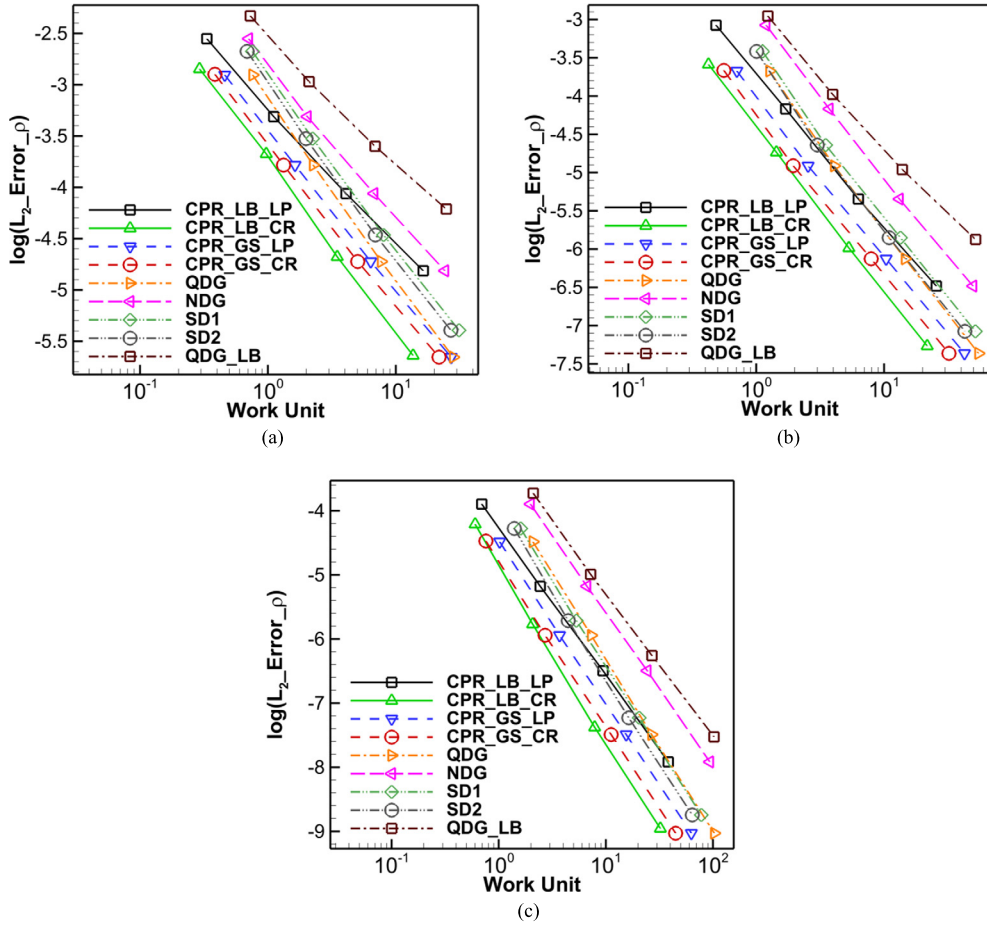


Fig. 4. L_2 error of the density versus work unit for the Euler vortex propagation problem using (a) P^2 , (b) P^3 and (c) P^4 reconstructions on meshes tessellated with regular linear elements.

- If a desired error level is given for any P^k solution reconstruction, the computational cost of the scheme CPR_LB_CR and the schemes from the CPR family with Gauss points are smaller than other schemes.
- In the CPR family, the CR based schemes are more computationally efficient than the LP based approach.
- Although the absolute errors of the schemes from the SD family are larger than that of QDG with the same order of accuracy, their numerical efficiency can be comparable to QDG, especially for higher degree reconstruction. This is largely due to the 1D based implementation of the SD scheme.

The connection between the work unit per DOF and the degree of the reconstruction polynomial for several schemes from CPR, QDG and SD families is displayed in Fig. 5. Interestingly, it is observed that in the test range of the reconstruction polynomial degrees (i.e., degree 2 to 4), for the schemes from both CPR and SD families, the cost per DOF will decrease as the reconstruction degree increases; for the schemes from the QDG family, the cost per DOF will increase as the reconstruction degree increases. These are closely related to the operations (i.e., matrix–vector product, interpolation and the Riemann solver) involved in these methods as previously discussed in Section 4.

To shed light on the different performance of QDG, SD and CPR, the computational cost of typical operations in the residual reconstruction of these formulations is briefly discussed. Three schemes, namely QDG, SD2 and CPR_GS_CR, are tested for the Euler vortex propagation case on an 80×80 mesh using P^4 reconstruction. The results are shown in Table 3. As the computational cost of Riemann solvers for all these schemes is the same, all other operations, e.g., interpolation, differentiation and matrix–vector multiplication, etc., are normalized by the computational cost of Riemann solvers. All the operations listed in this table are based on the implementation procedures discussed in Section 4. Note that ‘Surface Solution Preparation’ in Table 3 indicates the operations used to form the ‘left’ and ‘right’ states for the Riemann solver.

5.2.2. Error control performance on irregularly shaped bilinear elements

In this section, meshes tessellated with irregularly shaped bilinear elements are used to test the robustness of the aforementioned schemes. As the scheme QDG_LB does not achieve the optimal convergence rate even on regular linear

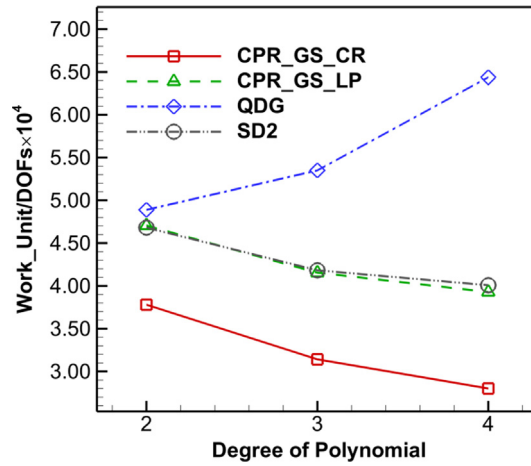


Fig. 5. Work unit per DOF versus the degree of reconstruction polynomial for the Euler vortex propagation problem using P^2 , P^3 and P^4 reconstructions with CPR_GS_CR, CPR_GS_LP, QDG and SD2 schemes on meshes tessellated with regular linear elements.

Table 3

Computational cost of major operations from QDG, SD2 and CPR_GS_CR. The simulations are carried out on an 80×80 mesh using P^4 reconstruction for the Euler vortex propagation problem. The computational time is computed from 1000 iterations.

QDG					
	Riemann solver	Volume integration	Surface integration	Surface solution preparation	
Relative time	1	6.59	4.30	1.03	
SD2					
	Riemann solver	Solution interpolation	Flux divergence	$\tilde{f}$ and $\tilde{g}$ calculation	Surface solution preparation
Relative time	1	1.85	2.37	1.48	0.86
CPR_GS_CR					
	Riemann solver	Solution gradient	Flux divergence	Flux correction	Surface solution and flux preparation
Relative time	1	1.38	0.61	1.69	1.19

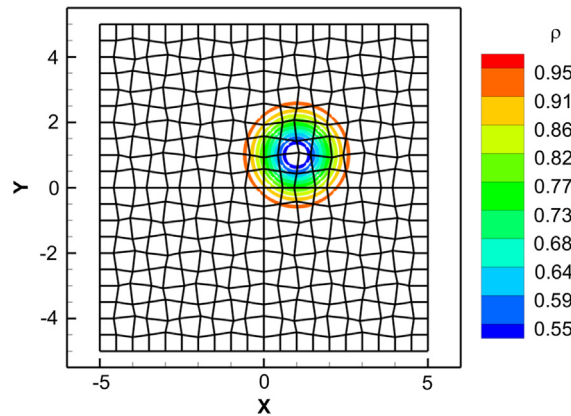


Fig. 6. The density field at $t = 1.0$ from numerical simulation with P^4 CPR_GS_LP reconstruction on a 20×20 mesh tessellated with irregularly shaped bilinear elements.

elements, this scheme will not be considered for further test. The density field of the propagating isentropic vortex on a distorted 20×20 grid with the P^4 CPR_GS_LP scheme at $t = 1.0$ is shown in Fig. 6. The convergence rates of P^2 to P^4 reconstructions for different schemes are presented in Fig. 7. It is found that the convergence rates of the schemes CPR_LB_LP and NDG are about half an order lower than the optimal one while the convergence rates of all other schemes achieve the optimal value. Based on further comparison of the convergence rate on irregular elements with that on regular elements as shown in Fig. 3, several observations are concluded as follows:

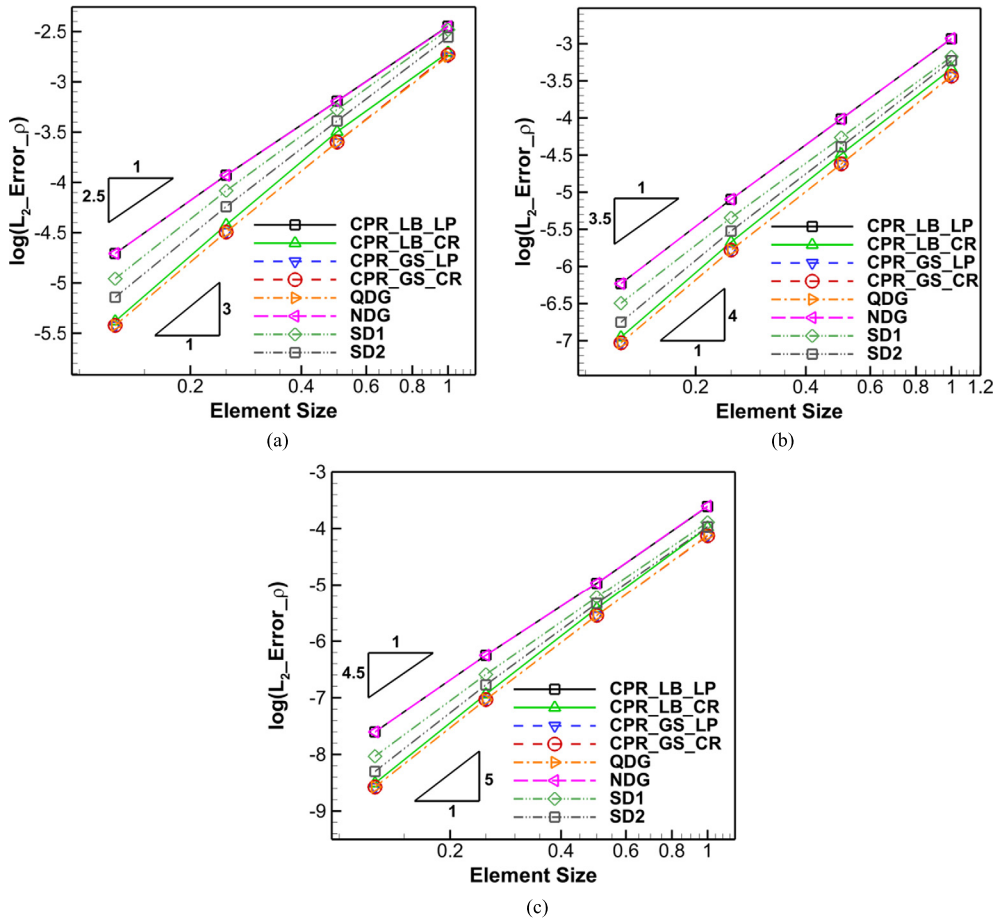


Fig. 7. Convergence test of the Euler vortex propagation problem for (a) P^2 , (b) P^3 and (c) P^4 reconstructions on meshes tessellated with irregularly shaped bilinear elements.

- For the two schemes from the SD family, although the convergence rate is the same, the absolute error of SD2 is smaller than that of SD1.
- On irregularly shaped bilinear elements, the scheme CPR_GS_LP is still equivalent to the scheme QDG, if the same set of solution (quadrature) points is used. Now, the scheme CPR_LB_LP is not equivalent to NDG, but the differences between these two schemes based on the present numerical tests are very small.

5.3. Accuracy and efficiency evaluation on curved elements

The same linear wave and isentropic vortex propagation cases as described in Section 5.2 are used here to test the performance of several schemes on curved elements. These schemes include the original and modified CPR_GS schemes, three types of QDG schemes and two types of SD schemes. For linear wave tests, the original and modified CPR_GS schemes are denoted as CPR_GS_Ori and CPR_GS_Mod respectively. For isentropic vortex propagation tests, schemes from the CPR family include CPR_GS_LP_Ori, CPR_GS_LP_Mod, CPR_GS_CR_Ori and CPR_GS_CR_Mod. As discussed in Section 4, schemes from the DG family include QDG, QDG_Q and QDG_H. The two schemes from the SD family are SD1 and SD2.

In this section, P^2 , P^3 and P^4 curved elements are adopted and the definition for these three types of curved elements is schematically shown in Fig. 8. Here, simplified curved elements are defined as in Refs. [33,42]. Lagrange shape functions associated with the vertex and edge points are used to build up all the metrics-related information of the curved element [50]. As stated in Section 3, the implementation of surface integrals in the QDG and modified CPR methods needs special attention. In order to ensure the accuracy of surface integrals, $(k+e)$ Gauss quadrature points are used to carry out surface integrals or form the surface mass matrices for a P^k curved element with P^k solution reconstruction.

5.3.1. Tests with one curved element

In order to better examine the performance of the aforementioned high-order schemes on curved elements, only one element is used in the test. Thus, the shape of the curved element can be well controlled and errors associated with the curved element can be clearly revealed.

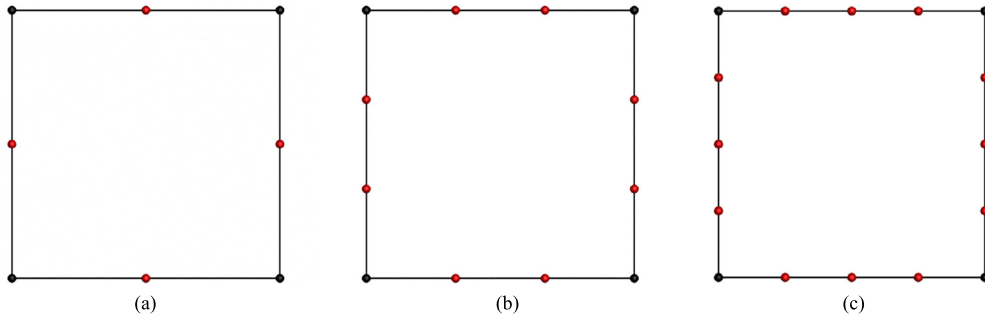


Fig. 8. Three types of high-order quadrilateral elements. (a) P^2 element; (b) P^3 element; (c) P^4 element.

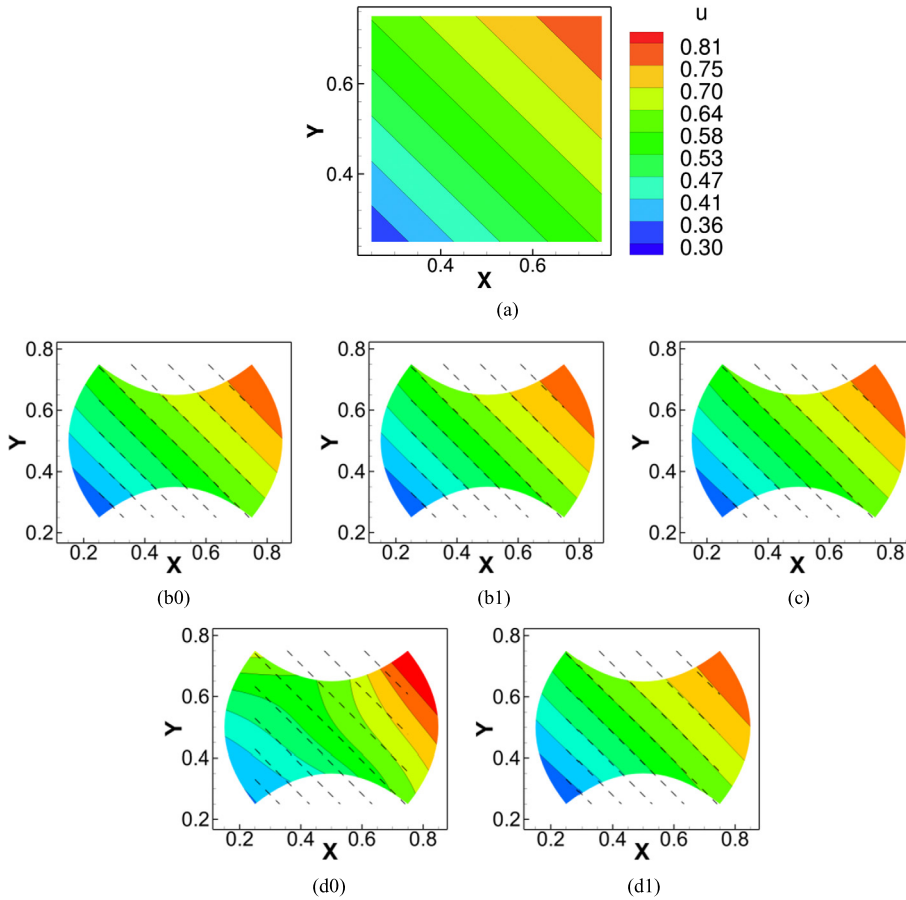


Fig. 9. u fields from the linear wave propagation test using one element. P^2 curved elements with 20% edge length deformation and P^2 solution reconstruction are used in (b0)–(d1). Dashed lines indicate the corresponding solution from (a) and all contour levels are kept the same as shown in (a). (a): Regular linear element with P^{10} CPR_GS_Ori scheme; (b0): CPR_GS_Ori scheme; (b1): CPR_GS_Mod scheme; (c): QDG scheme; (d0): SD1 scheme; (d1): SD2 scheme.

Tests for linear wave propagation First of all, the linear wave propagation problem is tested. The u field from the scheme CPR_GS_Ori at $t = 0.5$ for a P^{10} reconstruction on a regular element is displayed in Fig. 9(a) to serve as the benchmark of this case. Then u fields from P^2 solution reconstruction on one curved element are studied with several schemes from CPR, QDG and SD families. The shape of the element edge is controlled by a local cosine wave and fitted with a P^e ($e = 2, 3, 4$) polynomial. Here elements with 20% and 40% edge length deformation are used to examine the effects of element deformation on the schemes. Results on P^2 elements with 20% and 40% edge length deformation are displayed in Fig. 9(b0)–(d1) and Fig. 10. Then results on P^4 elements with 40% edge length deformation are shown in Fig. 11. Several observations can be summarized from the comparison of the presented linear cases.

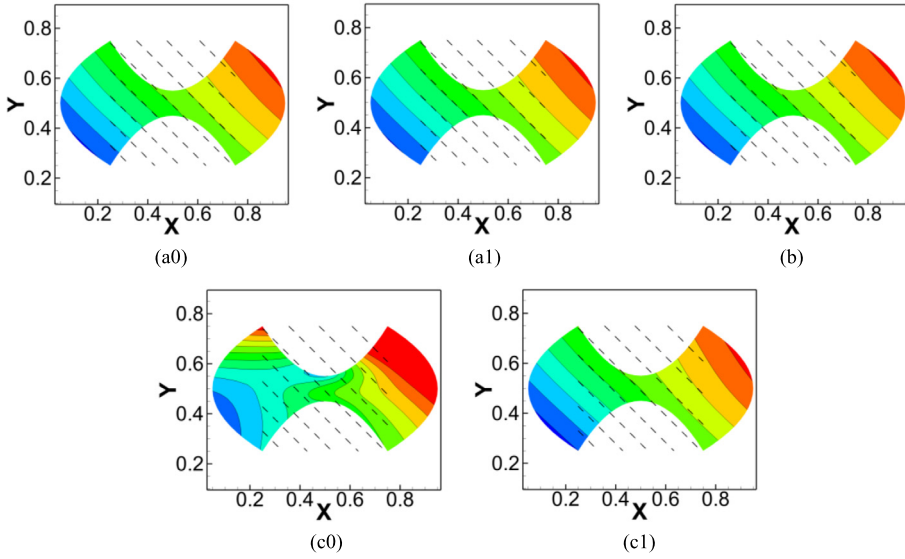


Fig. 10. u fields from the linear wave propagation test on one P^2 curved elements with 40% edge length deformation using P^2 (a0) CPR_GS_Ori scheme, (a1) CPR_GS_Mod scheme, (b) QDG scheme, (c0) SD1 scheme and (c1) SD2 scheme. Dashed lines indicate the corresponding solution from Fig. 9(a).

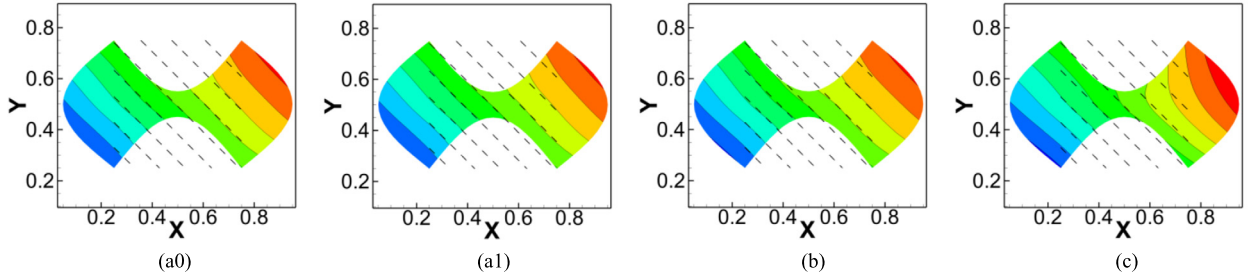


Fig. 11. u fields from the linear wave propagation test on one P^4 curved element with 40% edge length deformation using P^2 (a0) CPR_GS_Ori scheme, (a1) CPR_GS_Mod scheme, (b) QDG scheme and (c) SD2 scheme. Dashed lines indicate the corresponding solution from Fig. 9(a).

- For the same type of curved elements (e.g. P^2 element), as cell deformation becomes large, it is apparent that the absolute errors of the simulation results increase correspondingly for all schemes. Most schemes are not significantly affected by the edge curvature of the element, except for the SD1 scheme as shown in Fig. 9(d0) and Fig. 10(c0).
- The deteriorate performance of SD1 is mainly attributed to the fact that in this scheme $\tilde{Q}$ is assumed to belong to the solution space $\mathbb{Q}^k(\Omega_s)$. As mentioned previously, the nonlinear τ can drastically ruin the accuracy of the solution. When the assumption of $Q \in \mathbb{Q}^k(\Omega_s)$ is used, the accuracy of SD2 on curved elements can be recovered, as demonstrated by Fig. 9(d1) and Fig. 10(c1).
- With the same cell deformation (e.g. 40% edge length deformation), as the approximation degree of the curved element increases from P^2 to P^4 , the accuracy of schemes from CPR and QDG families does not apparently degrade. However, the nonlinear metrics from the flux polynomial $\tilde{F}_h^I$ can greatly affect the accuracy of schemes from the SD family, as demonstrated by the results from SD2 as shown in Fig. 10(c1) and Fig. 11(c).
- The modified scheme CPR_GS_Mod does not significantly improve the simulation accuracy on the highly curved element compared with the scheme CPR_GS_Ori. This indicates that for linear problems, as long as the assumption $Q \in \mathbb{Q}^k(\Omega_s)$ is held, the nonlinear τ and metrics will not drastically degrade the solution accuracy of the tensor-product-based schemes (i.e. CPR_GS_Ori and SD2), quadrature-based schemes (i.e. QDG) and mixed schemes (i.e. CPR_GS_Mod) on quadrilateral curved elements.

From the flow field comparison between SD1 and SD2 with P^9 solution reconstruction as shown in Fig. 12, it is found that if the degree of the solution space is sufficiently high, then almost no visible difference can be distinguished from the flow field simulated with the scheme assuming $Q \in \mathbb{Q}^k(\Omega_s)$ and that assuming $\tilde{Q} \in \mathbb{Q}^k(\Omega_s)$. From the p -convergence test on one element for the discussed schemes as shown in Fig. 13, several conclusions are drawn as follows:

- On the regular element, the schemes SD1 and SD2 are exactly the same. Although their absolute error is larger than that from the CPR and QDG families, the convergence rate is almost the same as that of the CPR and QDG families.

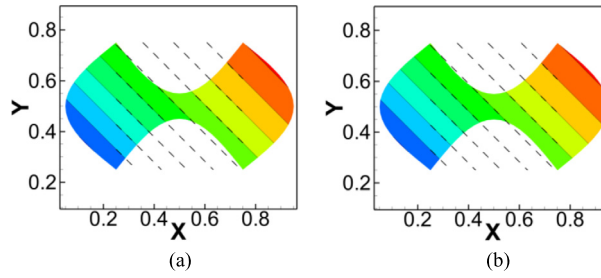


Fig. 12. u fields from the linear wave propagation test on one P^4 curved element with 40% edge length deformation using P^9 (a) SD1 and (b) SD2 schemes. Dashed lines in (a)–(d) indicate the corresponding solution from Fig. 9(a).

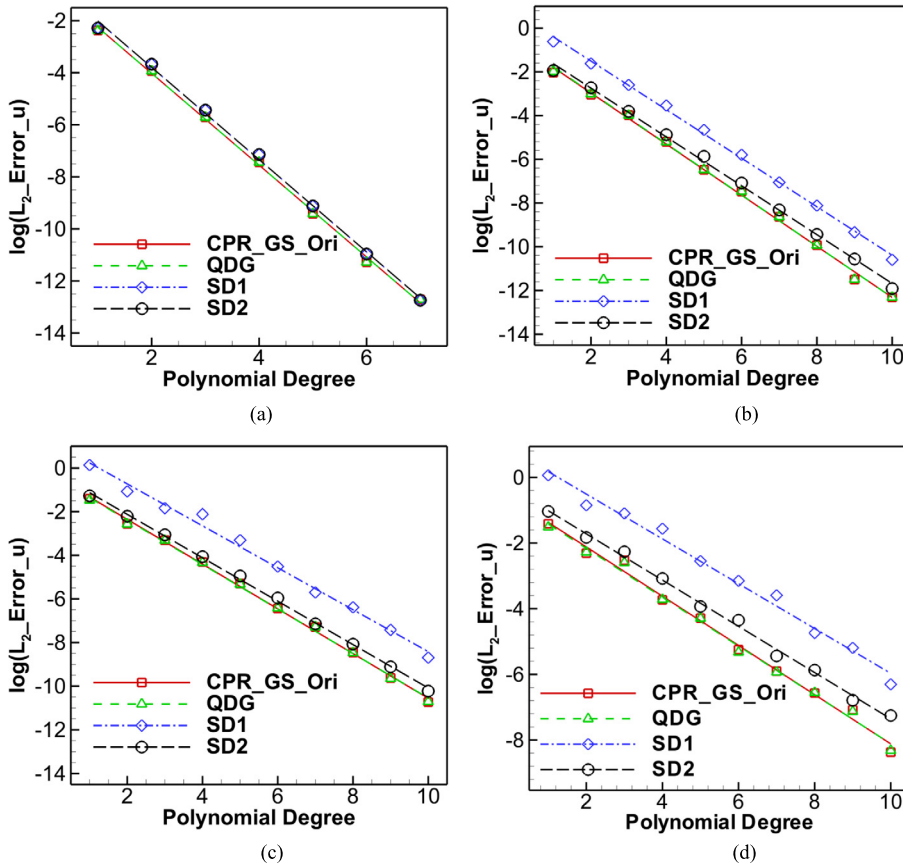


Fig. 13. p -convergence test of the linear wave propagation problem on one element with several schemes from CPR, DG and SD families. (a) Regular linear cell; (b) P^2 element with 20% edge length deformation; (c) P^2 element with 40% edge length deformation; (d) P^4 element with 40% edge length deformation.

- As also found in the previous example, higher polynomial degree of the curved element (e.g. P^4 element) and larger edge curvature (e.g. 40% edge length deformation) can degrade the solution accuracy and convergence rate, especially for the scheme SD1.

Tests for Euler vortex propagation Now we examine the results from the isentropic vortex propagation. The density field from the scheme CPR_GS_LP_Ori at $t = 0.5$ for a P^{10} reconstruction on a regular element is displayed in Fig. 14(a) to serve as the benchmark of this case. Then density fields on one P^2 element with 20% and 40% edge length deformation for several schemes from CPR, QDG and SD families are displayed in Fig. 14(b0)–(d) and Fig. 15, respectively. Here, P^2 solution reconstruction is used for all tests. From the comparison with results from the linear conservation law, several observations are concluded as follows:

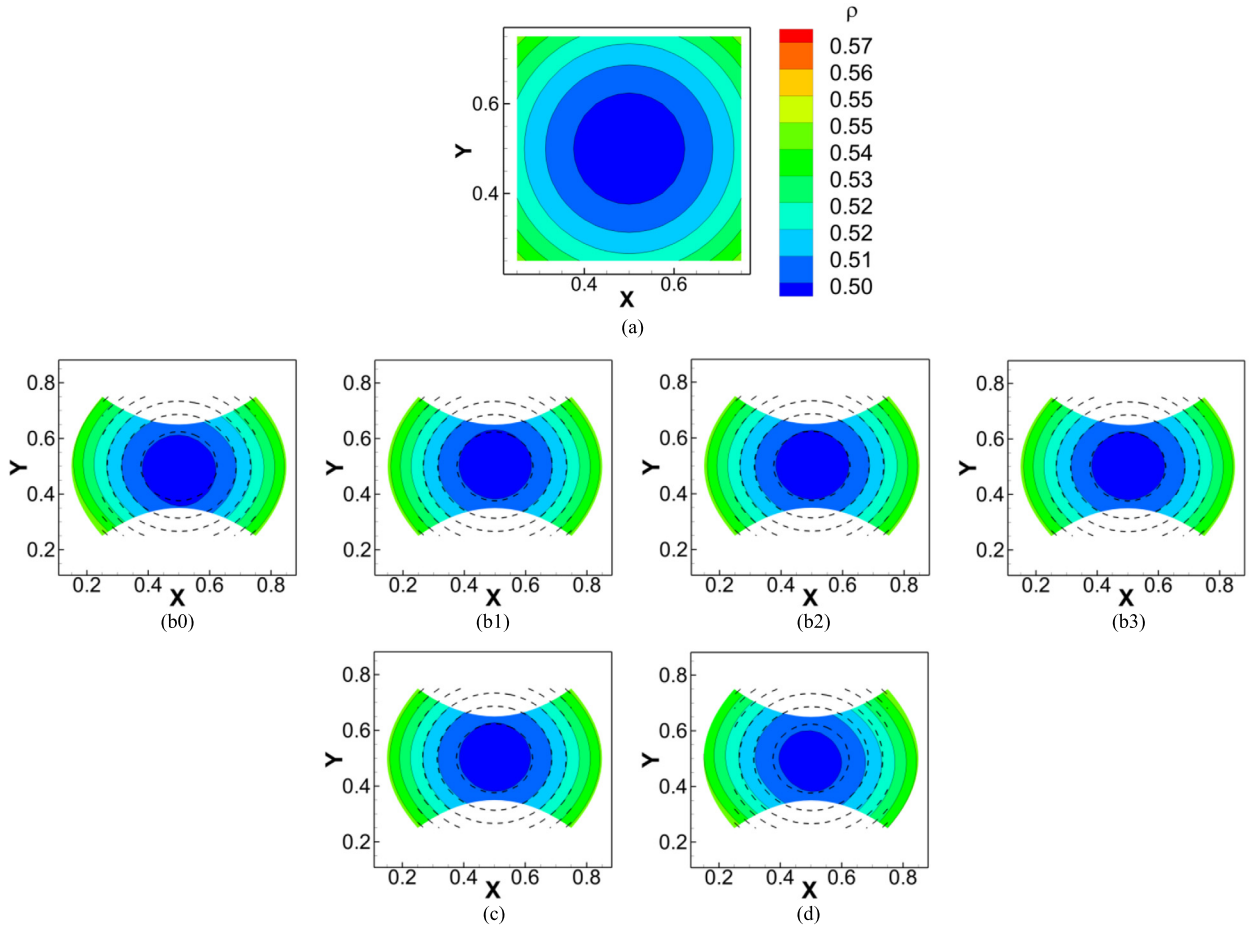


Fig. 14. Density fields from the Euler vortex propagation test on one element. P^2 curved elements with 20% edge length deformation and P^2 solution reconstruction are used in (b0)–(d). Dashed lines indicate the corresponding solution from (a) and all contour levels are kept the same as shown in (a). (a): Regular linear element with P^{10} CPR_GS_LP_Ori scheme; (b0): CPR_GS_LP_Ori scheme; (b1) CPR_GS_LP_Mod scheme; (b2): CPR_GS_CR_Ori scheme; (b3): CPR_GS_CR_Mod scheme; (c): QDG scheme; (d): SD2 scheme.

- For the Lagrange polynomial approach based CPR schemes, if the element deformation is not severe, then both the original tensor-product-based scheme (i.e. CPR_GS_LP_Ori) and the mixed scheme (i.e. CPR_GS_LP_Mod) work well on the curved element. However, if the element deformation is severe, then the accuracy of the scheme CPR_GS_LP_Ori degrades noticeably while the modified scheme CPR_GS_LP_Mod can recover the accuracy.
- The performance of the chain-rule approach based CPR schemes (i.e. CPR_GS_CR_Ori and CPR_GS_CR_Mod) is surprisingly good on curved elements with both small and large edge curvature.
- The scheme QDG (or QDG_Q) works well on curved elements with both small and large edge curvature. This is due to the fact that the quadrature rule for all integration is carried out with sufficient number of quadrature points.
- The accuracy of the scheme SD2 is adversely affected by the curved element, even with small edge curvature.

In order to further clarify the influence of the approximation degree of the curved element on nonlinear conservation laws, density fields of the isentropic vortex propagation case from several schemes with P^4 solution reconstruction on P^2 and P^4 elements are displayed in Fig. 16. 40% edge length deformation is enforced on the element. From the results using the scheme CPR_GS_LP_Ori as shown in Fig. 16(a) and (b), it is observed that with the same solution reconstruction, the density field on the high-order (P^4) element departs from the exact solution more than that on the low-order (P^2) element due to the higher degree τ and metrics. The results from CPR_GS_LP_Ori (Fig. 16(b)) and QDG (Fig. 16(c)) on the P^4 element are very similar. When more quadrature points are used for volume and surface integration in QDG, the density field on the P^4 element is almost the same as that on the regular element, as shown in Fig. 16(d) for QDG_H.

All test results from P^2 and P^4 solution reconstruction on one P^2 , P^3 or P^4 element with different schemes for the Euler vortex propagation problem are tabulated in Tables 4 and 5. Herein, ‘Case1’ indicates that the curved element has 20% edge length deformation, while ‘Case2’ indicates that the curved element has 40% edge length deformation. The data in Tables 4 and 5 can further confirm the conclusions drawn from the previous analysis. Besides, it is also found that the scheme QDG_H

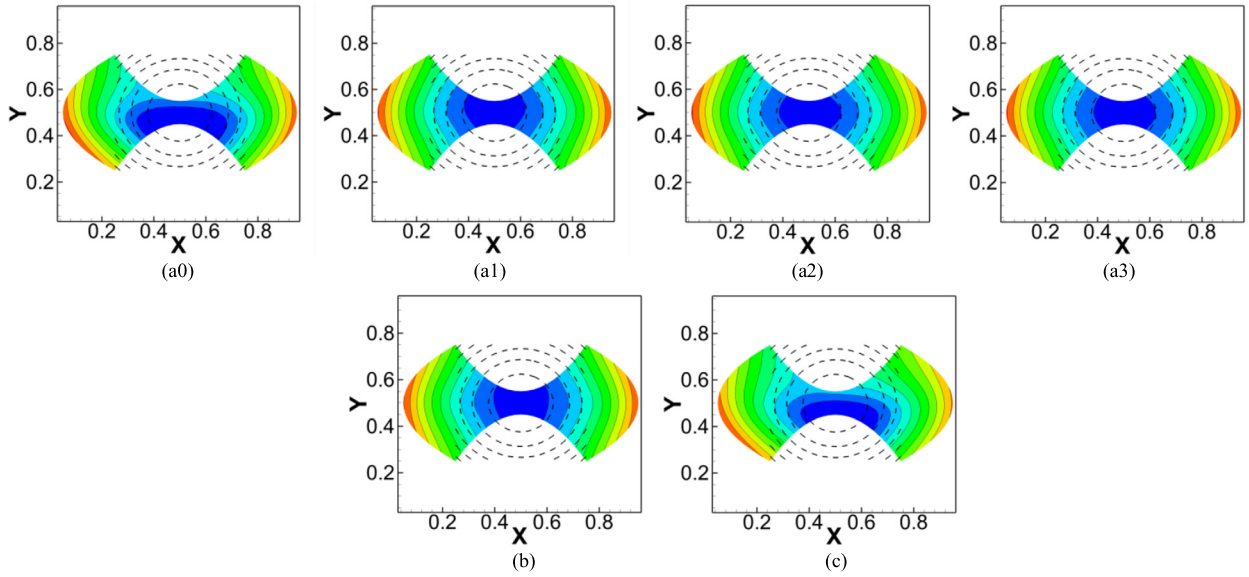


Fig. 15. Density fields from the Euler vortex propagation test on one P^2 curved elements with 40% edge length deformation using P^2 (a0) CPR_GS_LP_Ori scheme, (a1) CPR_GS_LP_Mod scheme, (a2) CPR_GS_CR_Ori scheme, (a3) CPR_GS_CR_Mod scheme, (b) QDG scheme and (c) SD2 scheme. Dashed lines indicate the corresponding solution from Fig. 14(a).

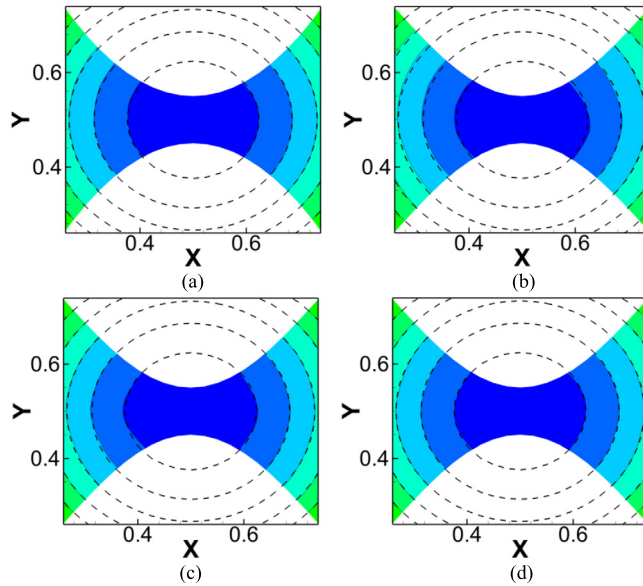


Fig. 16. Density fields from the one curved element Euler vortex propagation test with P^4 solution reconstruction. (a) P^2 element with CPR_GS_LP_Ori; (b) P^4 element with CPR_GS_LP_Ori; (c) P^4 element with QDG; (d) P^4 element with QDG_H. Dashed lines in (a)–(d) indicate the corresponding solution from Fig. 14(a).

can help enhance the scheme accuracy (decreasing the nonlinear aliasing errors) especially for the higher-order (e.g., P^4) reconstruction. Then p -convergence tests for several typical schemes on one curved element but with different geometry approximation degrees and edge curvature are carried out and the results are shown in Fig. 17. Similar conclusions can be drawn as those from Fig. 13.

5.3.2. Tests with mesh tessellated with curved elements

In this section, accuracy and efficiency of the schemes aforementioned are examined on a mesh tessellated with curved elements. The shape of the element is controlled by sinusoidal waves. Specifically, the curved mesh (x, y) can be obtained from the corresponding Cartesian mesh $(x_{\text{Cart}}, y_{\text{Cart}})$ as follows

Table 4

Euler vortex propagation test on one curved element with P^2 solution reconstruction. 'Case1' indicates that the curved element has 20% edge length deformation; 'Case2' indicates that the curved element has 40% edge length deformation.

		CPR_GS_LP_Ori	CPR_GS_LP_Mod	CPR_GS_CR_Ori	CPR_GS_CR_Mod
P^2 elem.	Case1	1.024×10^{-3}	5.147×10^{-4}	5.246×10^{-4}	5.033×10^{-4}
	Case2	6.172×10^{-3}	1.803×10^{-3}	1.996×10^{-3}	1.888×10^{-3}
P^3 elem.	Case1	9.636×10^{-4}	5.056×10^{-4}	5.076×10^{-4}	4.927×10^{-4}
	Case2	5.685×10^{-3}	1.711×10^{-3}	1.873×10^{-3}	1.780×10^{-3}
P^4 elem.	Case1	1.402×10^{-3}	1.374×10^{-3}	1.433×10^{-3}	1.280×10^{-3}
	Case2	5.583×10^{-3}	2.667×10^{-3}	3.534×10^{-3}	2.505×10^{-3}
		QDG	QDG_Q	QDG_H	SD1
P^2 elem.	Case1	5.997×10^{-4}	5.997×10^{-4}	6.177×10^{-4}	1.034×10^{-2}
	Case2	1.923×10^{-3}	1.923×10^{-3}	1.902×10^{-3}	8.152×10^{-2}
P^3 elem.	Case1	5.896×10^{-4}	5.896×10^{-4}	6.054×10^{-4}	9.780×10^{-3}
	Case2	1.853×10^{-3}	1.853×10^{-3}	1.838×10^{-3}	7.410×10^{-2}
P^4 elem.	Case1	1.133×10^{-3}	1.133×10^{-3}	1.150×10^{-3}	2.292×10^{-2}
	Case2	2.237×10^{-3}	2.237×10^{-3}	2.222×10^{-3}	8.619×10^{-2}
		SD2			
P^2 elem.	Case1	1.665×10^{-3}			
	Case2	6.899×10^{-3}			
P^3 elem.	Case1	1.601×10^{-3}			
	Case2	6.379×10^{-3}			
P^4 elem.	Case1	3.169×10^{-3}			
	Case2	9.004×10^{-3}			

Table 5

Euler vortex propagation test on one curved element with P^4 solution reconstruction. 'Case1' indicates that the curved element has 20% edge length deformation; 'Case2' indicates that the curved element has 40% edge length deformation.

		CPR_GS_LP_Ori	CPR_GS_LP_Mod	CPR_GS_CR_Ori	CPR_GS_CR_Mod
P^2 elem.	Case1	2.524×10^{-5}	2.275×10^{-5}	2.138×10^{-5}	2.195×10^{-5}
	Case2	1.390×10^{-4}	1.056×10^{-4}	1.116×10^{-4}	9.173×10^{-5}
P^3 elem.	Case1	2.398×10^{-5}	2.201×10^{-5}	2.033×10^{-5}	2.117×10^{-5}
	Case2	1.291×10^{-4}	1.005×10^{-4}	1.036×10^{-4}	8.793×10^{-5}
P^4 elem.	Case1	6.749×10^{-5}	5.985×10^{-5}	3.796×10^{-5}	3.550×10^{-5}
	Case2	4.124×10^{-4}	2.764×10^{-4}	2.243×10^{-4}	1.855×10^{-4}
		QDG	QDG_Q	QDG_H	SD1
P^2 elem.	Case1	2.282×10^{-5}	2.282×10^{-5}	2.172×10^{-5}	1.004×10^{-4}
	Case2	1.124×10^{-4}	1.124×10^{-4}	9.969×10^{-5}	1.390×10^{-3}
P^3 elem.	Case1	2.204×10^{-5}	2.204×10^{-5}	2.084×10^{-5}	9.241×10^{-5}
	Case2	1.055×10^{-4}	1.055×10^{-4}	9.407×10^{-5}	1.242×10^{-3}
P^4 elem.	Case1	5.732×10^{-5}	5.732×10^{-5}	3.165×10^{-5}	1.062×10^{-3}
	Case2	2.504×10^{-4}	2.504×10^{-4}	1.795×10^{-4}	6.144×10^{-3}
		SD2			
P^2 elem.	Case1	6.529×10^{-5}			
	Case2	4.243×10^{-4}			
P^3 elem.	Case1	6.117×10^{-5}			
	Case2	3.957×10^{-4}			
P^4 elem.	Case1	2.048×10^{-4}			
	Case2	9.558×10^{-4}			

$$\begin{cases} x = x_{Cart} + A_x \sin(T_x y_{Cart}), \\ y = y_{Cart} + A_y \sin(T_y x_{Cart}). \end{cases} \quad (41)$$

For all meshes used in the tests, the following parameters are specified. $A_x = 0.4$, $A_y = 0.2$, $T_x = 0.8\pi$ and $T_y = \pi$. A fine mesh with 80×80 elements and an enlarged view of the distorted mesh are displayed in Fig. 18(a) and (b). The u field from the linear wave propagation test and the density field of the propagating isentropic vortex on a distorted 20×20 grid with the P^4 CPR_GS_Ori and CPR_GS_LP_Ori schemes at $t = 1.0$ are shown in Fig. 18(c) and (d), respectively. The convergence rates on meshes tessellated with P^2 and P^4 elements for P^2 to P^4 solution reconstruction with several typical schemes from CPR, QDG and SD families are displayed in Fig. 19. It is clear that all tested schemes can preserve the optimal convergence rate on meshes tessellated with curved elements. All tested schemes from CPR and QDG families achieve lower absolute errors than the schemes from the SD family. The absolute errors with respect to the work units for tested schemes with P^4 solution reconstruction on meshes consisting of P^2 and P^4 elements are displayed in Fig. 20. Several observations are summarized as follows:

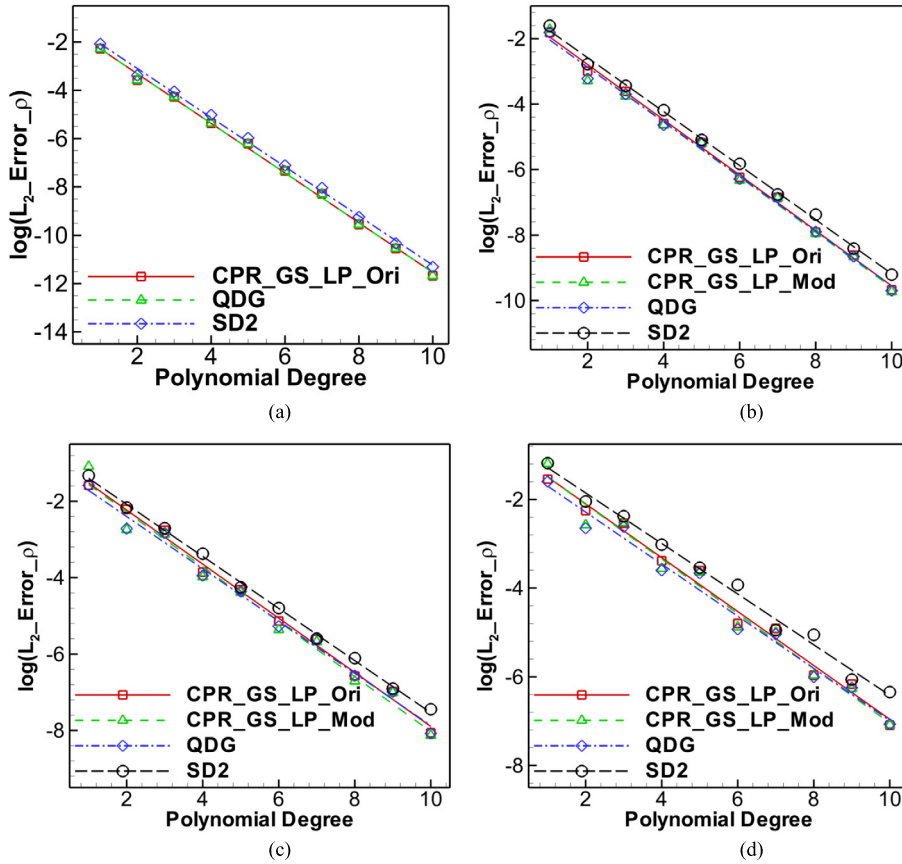


Fig. 17. p -convergence test of the Euler vortex propagation problem on one element with several schemes from CPR, DG and SD families. (a) Regular linear cell; (b) P^2 element with 20% edge length deformation; (c) P^2 element with 40% edge length deformation; (d) P^4 element with 40% edge length deformation.

- The absolute errors from the schemes QDG and QDG_Q are the same, but the computational cost of QDG is less than that of QDG_Q. This validates the formation process of the surface matrix $\tilde{B}_f$ on curved elements developed in Section 3.1. Meanwhile, this test also confirms that the quadrature-free implementation can be used to save computational cost.
- The tensor-product-based CPR schemes, namely CPR_GS_LP_Ori and CPR_GS_CR_Ori, are most cost-effective when a certain error level is achieved.
- Despite the fact that the absolute error of the scheme SD2 is larger than the schemes from CPR and QDG, it is still cost-effective when a certain error level is achieved.
- Although the scheme QDG_H can help reduce the aliasing errors for the nonlinear conservation law, it is much more expensive than other schemes based on the test results.

Note that the mixed schemes, e.g., CPR_GS_LP_Mod, do not outperform their original counterparts for the present test cases. This is possibly due to the fact that the deformation of the curved element is not as severe as that of the one element case presented in Section 5.3.1.

5.3.3. Inviscid flow over the NACA0012 airfoil

Finally, accuracy and efficiency of several schemes are tested with inviscid flow over the NACA0012 airfoil. Two sets of meshes tessellated with quadrilateral elements from the 1st International Workshop on High-Order CFD Methods [35] are used in the test. There are 140 elements in the coarse mesh and 8960 elements in the fine one. Far field boundaries for both meshes are located more than 2000 chord length away from the airfoil. Note that solid wall boundaries are represented with high-order (P^4) polynomials and correspondingly high-order curved elements are used near the solid wall. Bilinear quadrilateral elements are used elsewhere in the mesh. Based on the current setup of element types, the simulations using schemes from the CPR family adopt the following strategy. The original CPR schemes are applied on the irregularly shaped bilinear elements for all cases; both the original and modified CPR schemes are tested on high-order curved elements near the solid wall. The implementation of QDG scheme automatically distinguishes different element types by design. The P^3 solution reconstruction is used for space discretization. The GMRES algorithm [51] with a block LU-SGS [52] preconditioner

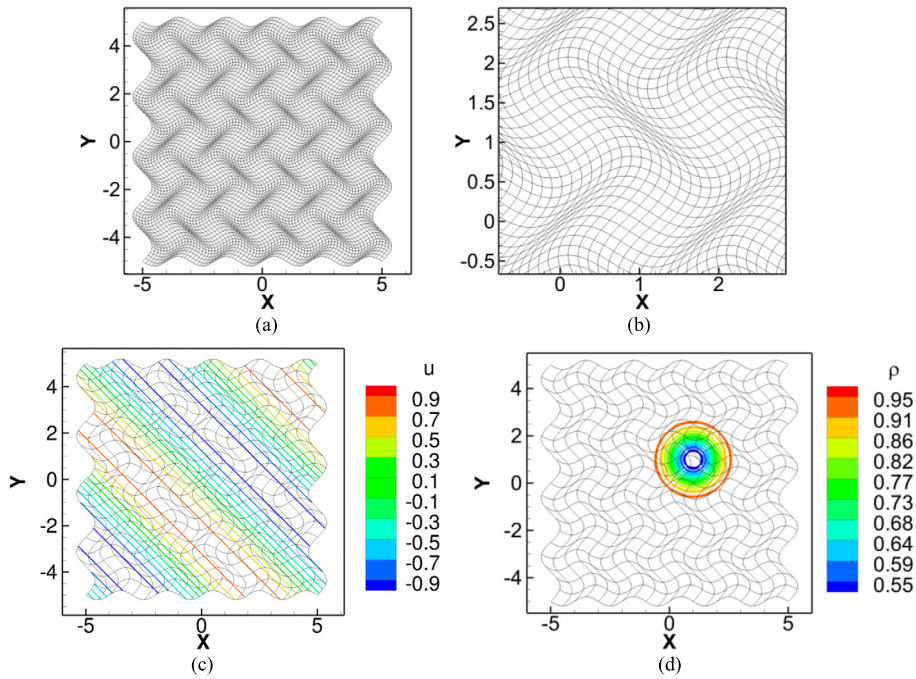


Fig. 18. Meshes tessellated with curved elements for convergence study. (a) Overview of an 80×80 mesh; (b) enlarged view near highly distorted mesh in (a); (c) u field from the linear wave propagation test on a 20×20 mesh; (d) density field from the Euler vortex propagation test on a 20×20 mesh.

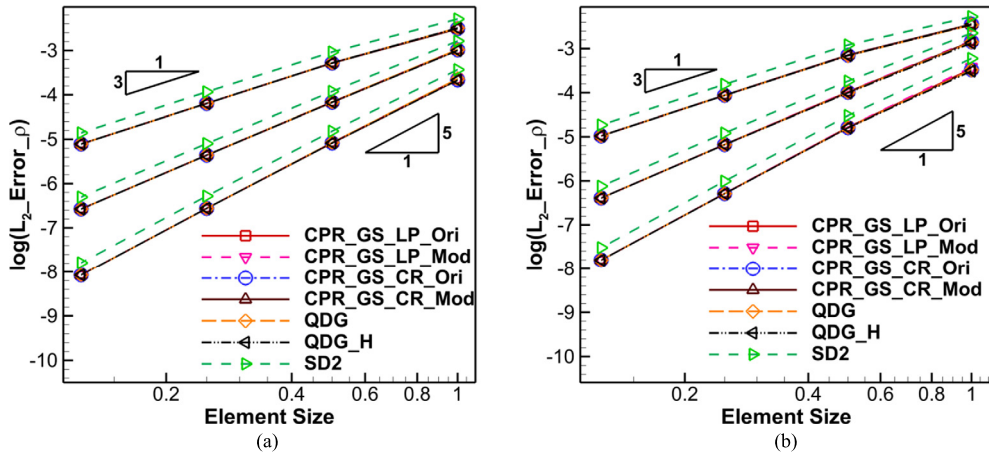


Fig. 19. Convergence test of the Euler vortex propagation problem for P^2 , P^3 and P^4 reconstructions on meshes tessellated with (a) P^2 curved elements and (b) P^4 curved elements.

is used for time integration. The flow conditions are specified as follows. The Mach number on far field boundaries is 0.5 and the angle of attack of the incoming flow is 0° .

The converged Mach number fields on both coarse and fine grids are displayed in Fig. 21. Note that the accuracy performance for all schemes tested is quite similar. Therefore, only results from the CPR_GS_CR_Ori scheme are displayed in this figure. The convergence paths with respect to the iteration steps and work unit for these schemes on both coarse and fine grids are shown in Fig. 22. From this figure, it is found that CR based CPR schemes, namely CPR_LB_CR_Ori and CPR_GS_CR_Ori are most cost-effective among all schemes tested based on the current convergence criteria (namely, errors drop to machine zero, i.e., $\sim O(10^{-16})$). The performances of CPR_GS_LP_Ori, CPR_GS_LP_Mod and QDG are comparable.

6. Conclusions

Numerical accuracy and efficiency of several discontinuous high-order methods, including QDG, NDG, SD and CPR, for the conservation laws are analyzed and compared on both linear and curved quadrilateral elements. A quadrature-free approach is developed for surface integration on curved elements to reduce the computational cost of the existing quadrature-based

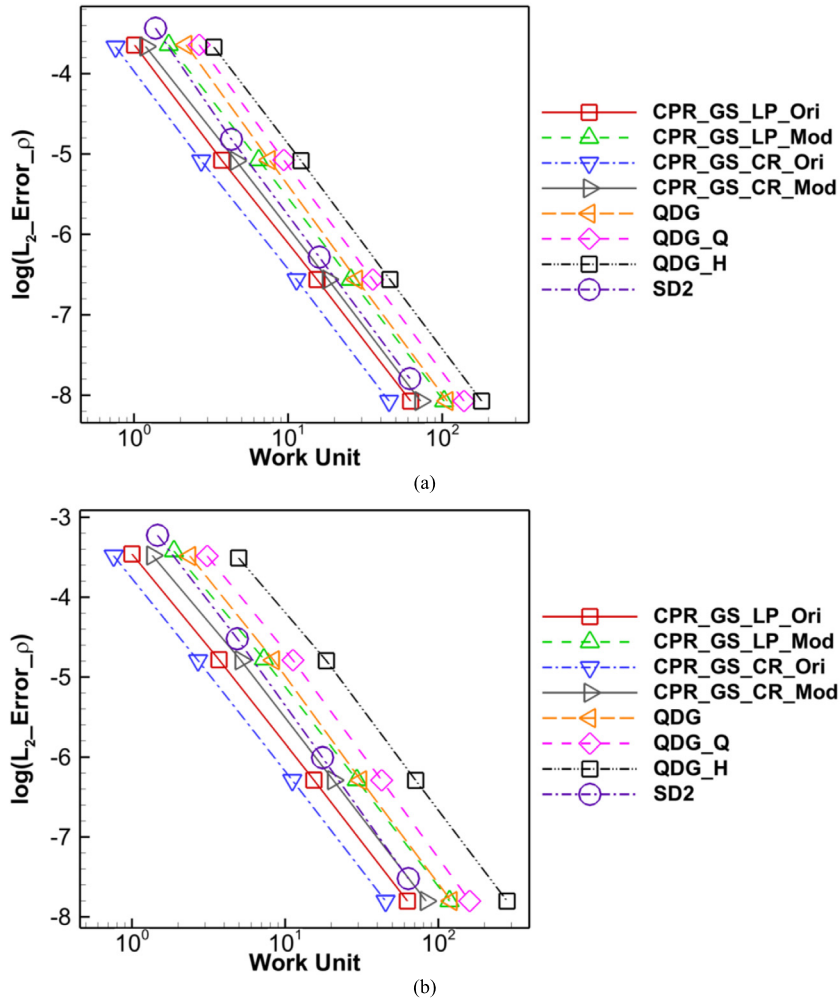


Fig. 20. L_2 error of the density versus work unit for the Euler vortex propagation problem using P^4 reconstructions on meshes tessellated with (a) P^2 curved elements and (b) P^4 curved elements.

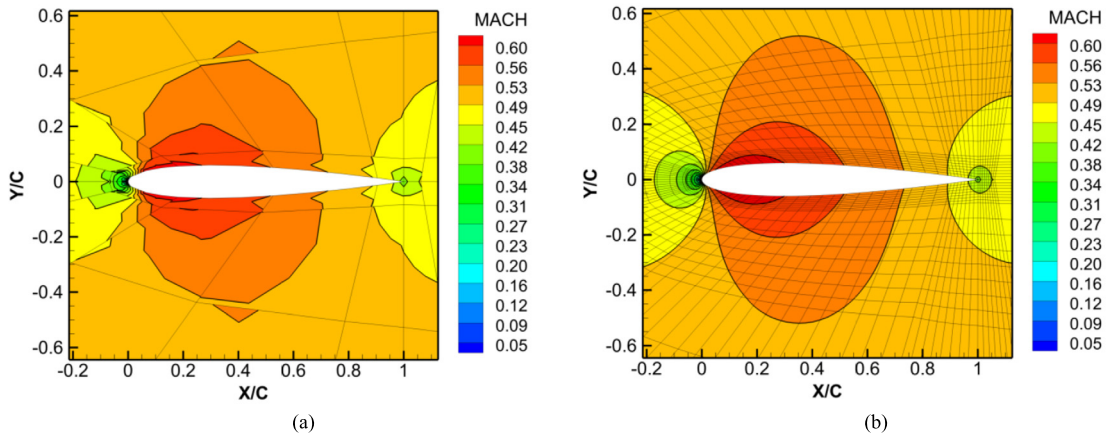


Fig. 21. Mach number fields over the NACA0012 airfoil on both (a) coarse and (b) fine grids using the P^3 CPR_GS_CR_Ori scheme. The inlet flow Mach number is 0.5 and the angle of attack is 0° .

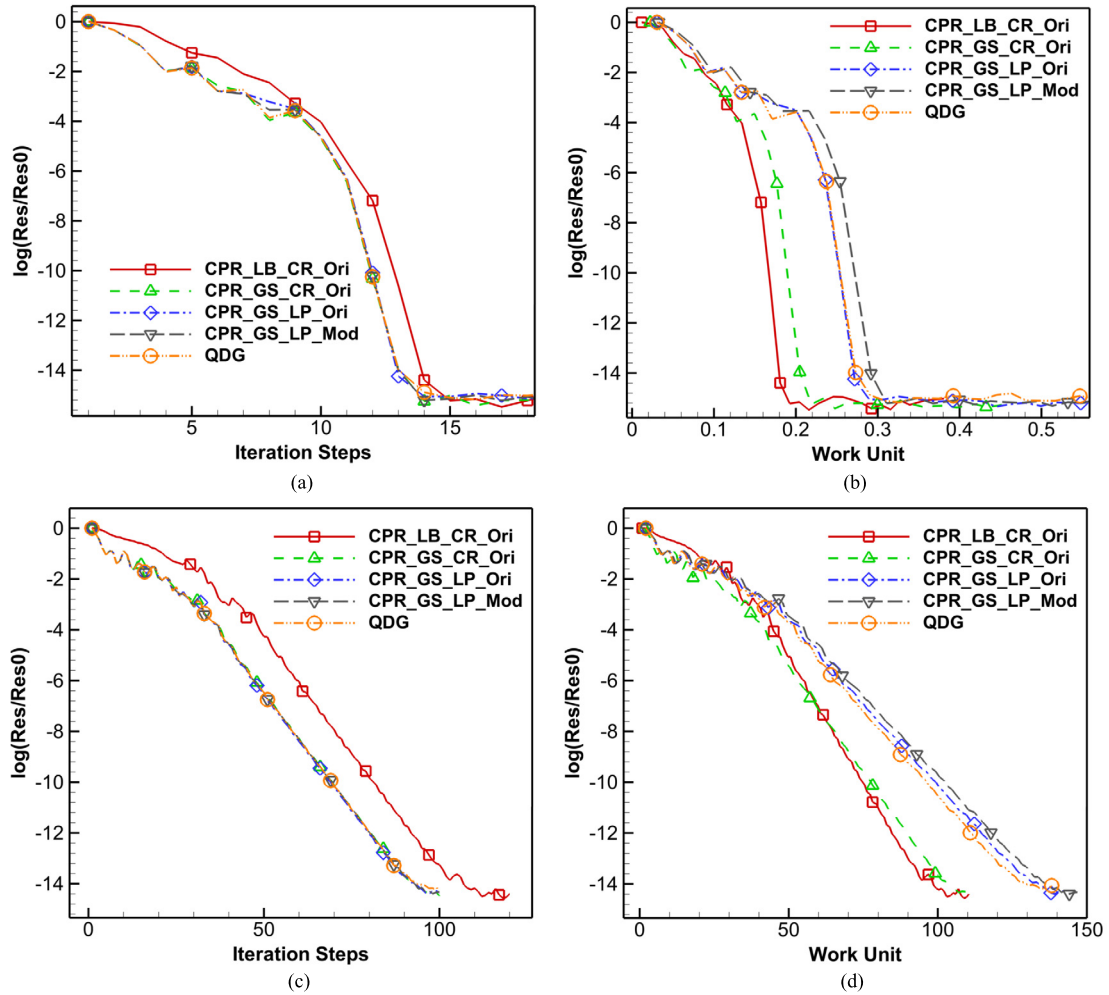


Fig. 22. The convergence paths with respect to the iteration steps and work unit on both (a), (c) coarse and (b), (d) fine grids for several schemes.

DG without losing accuracy. In view of the observations from the numerical experiments, a new mixed type of CPR and DG scheme for highly skewed element is proposed to enhance the accuracy of the LP approach based CPR schemes. It turns out that the accuracy and efficiency of the newly developed CPR scheme is similar to QDG using the quadrature-free approach on both linear and curved element.

Based on the test results, it is found that most schemes from the CPR family cost less work units to achieve a desired error level than other high-order methods on both linear and curved elements. The CR approach based CPR schemes work surprisingly well even on highly skewed elements. All these features demonstrate that the CPR scheme is a competitive algorithm to solve the conservation law on quadrilateral elements. Although the absolute error of the SD scheme is larger than that of the CPR and QDG schemes, the SD scheme is still a viable candidate of cost-effective schemes. Based on the current numerical experiments, it is suggested that for flow simulations using quadrilateral elements, if the deformation of curved elements is not severe, tensor-product-based schemes are cost-effective and should be the first choice. The mixed type scheme is a good trade-off between accuracy and computational cost on highly skewed element.

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References

- [1] G.E. Karniadakis, S.J. Sherwin, *Spectral/hp Element Methods for Computational Fluid Dynamics*, Oxford University Press, Oxford, 2005.
- [2] J.S. Hesthaven, T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*, Springer-Verlag, New York, 2008.

- [3] Z.J. Wang, Adaptive High-Order Methods in Computational Fluid Dynamics, *Advances in Computational Fluid Dynamics*, vol. 2, World Scientific, 2011.
- [4] J.A. Ekaterinaris, High-order accurate, low numerical diffusion methods for aerodynamics, *Prog. Aerosp. Sci.* 41 (2005) 192–300.
- [5] Z.J. Wang, High-order methods for the Euler and Navier–Stokes equations on unstructured grids, *Prog. Aerosp. Sci.* 43 (2007) 1–47.
- [6] W.H. Reed, T.R. Hill, Triangular mesh methods for the neutron transport equation, Los Alamos Scientific Laboratory Report, 1973.
- [7] B. Cockburn, C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework, *Math. Comput.* 52 (1989) 411–435.
- [8] B. Cockburn, C.-W. Shu, The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems, *J. Comput. Phys.* 141 (1998) 199–224.
- [9] B. Cockburn, S.Y. Lin, C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one-dimensional systems, *J. Comput. Phys.* 84 (1989) 90–113.
- [10] F. Bassi, S. Rebay, High-order accurate discontinuous finite element solution of the 2D Euler equations, *J. Comput. Phys.* 138 (1997) 251–285.
- [11] F. Bassi, S. Rebay, A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier–Stokes equations, *J. Comput. Phys.* 131 (1997) 267–279.
- [12] B. Cockburn, C.-W. Shu, Runge–Kutta discontinuous Galerkin methods for convection-dominated problems, *J. Sci. Comput.* 16 (2001) 173–261.
- [13] D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.* 39 (2002) 1749–1779.
- [14] H.L. Atkins, C.-W. Shu, Quadrature-free implementation of discontinuous Galerkin method for hyperbolic equations, *AIAA J.* 36 (1998) 775–782.
- [15] G. Cohen, M. Duruflé, Non spurious spectral-like element methods for Maxwell's equations, *J. Comput. Math.* 25 (3) (2007) 282–304.
- [16] D.A. Kopriva, S.L. Woodruff, M.Y. Hussaini, Discontinuous spectral element approximation of Maxwell's equations, in: *Discontinuous Galerkin Methods: Theory, Computation and Applications*, Springer-Verlag, New York, 2000, pp. 355–362.
- [17] D.A. Kopriva, *Implementing Spectral Methods for Partial Differential Equations: Algorithms*, Springer-Verlag, Berlin, 2009.
- [18] D. Kopriva, G. Gassner, On the quadrature and weak form choices in collocation type discontinuous Galerkin spectral element methods, *J. Sci. Comput.* 44 (2) (2010) 136–155.
- [19] D.A. Kopriva, J.H. Kolias, A conservative staggered-grid Chebyshev multidomain method for compressible flows, *J. Comput. Phys.* 125 (1996) 244–261.
- [20] Y. Liu, M. Vinokur, Z.J. Wang, Spectral difference method for unstructured grids I: Basic formulation, *J. Comput. Phys.* 216 (2006) 780–801.
- [21] G. May, A. Jameson, A spectral difference method for the Euler and Navier–Stokes equations, *AIAA Paper* 2006-304.
- [22] H.T. Huynh, A flux reconstruction approach to high-order schemes including discontinuous Galerkin methods, *AIAA Paper* 2007-4079.
- [23] K.V.d. Abeele, C. Lacor, Z.J. Wang, On the stability and accuracy of the spectral difference method, *J. Sci. Comput.* 37 (2008) 162–188.
- [24] Z.J. Wang, Spectral (finite) volume method for conservation laws on unstructured grids: basic formulation, *J. Comput. Phys.* 178 (2002) 210–251.
- [25] H.T. Huynh, A reconstruction approach to high-order schemes including discontinuous Galerkin for diffusion, *AIAA Paper* 2009-403.
- [26] Z.J. Wang, H.Y. Gao, A unifying lifting collocation penalty formulation for the Euler equations, *AIAA Paper* 2009-401.
- [27] Z.J. Wang, H.Y. Gao, A unifying lifting collocation penalty formulation including the discontinuous Galerkin, spectral volume/difference methods for conservation laws on mixed grids, *J. Comput. Phys.* 228 (2009) 8161–8186.
- [28] H.Y. Gao, Z.J. Wang, A high-order lifting collocation penalty formulation for the Navier–Stokes, *AIAA Paper* 2009-3784.
- [29] T. Haga, H.Y. Gao, Z.J. Wang, A high-order unifying discontinuous formulation for the Navier–Stokes equations on 3D mixed grids, *Math. Model. Nat. Phenom.* 6 (2011) 28–56.
- [30] H.T. Huynh, High-order methods including discontinuous Galerkin by reconstructions on triangular meshes, *AIAA Paper* 2011-44.
- [31] M.L. Yu, Z.J. Wang, On the connection between the correction and weighting functions in the correction procedure via reconstruction method, *J. Sci. Comput.* 54 (1) (2013) 227–244.
- [32] J. Cagnone, S. Nadarajah, A stable interface element scheme for the p -adaptive lifting collocation penalty formulation, *J. Comput. Phys.* 231 (2012) 1615–1634.
- [33] J. Cagnone, B. Vermeire, S. Nadarajah, A p -adaptive LCP formulation for the compressible Navier–Stokes equations, *J. Comput. Phys.* 233 (2013) 324–338.
- [34] P.O. Persson, A sparse and high-order accurate line-based discontinuous Galerkin method for unstructured meshes, *J. Comput. Phys.* 233 (2013) 414–429.
- [35] Z.J. Wang, K. Fidkowski, R. Abgrall, F. Bassi, D. Caraeni, A. Cary, H. Deconinck, R. Hartmann, K. Hillewaert, H. Huynh, N. Kroll, G. May, P.-O. Persson, B.V. Leer, M. Visbal, High-order CFD methods: current status and perspective, *Int. J. Numer. Methods Fluids* 72 (8) (2013) 811–845.
- [36] P.L. Roe, Approximate Riemann solvers, parameter vectors and difference schemes, *J. Comput. Phys.* 43 (2) (1981) 357–372.
- [37] Y.Z. Sun, Z.J. Wang, Y. Liu, High-order multidomain spectral difference method for the Navier–Stokes equations on unstructured hexahedral grids, *Commun. Comput. Phys.* 2 (2) (2006) 310–333.
- [38] A. Jameson, P.E. Vincent, P. Castonguay, On the non-linear stability of flux reconstruction schemes, *J. Sci. Comput.* 50 (2012) 434–445.
- [39] H.Y. Gao, Z.J. Wang, A conservative correction procedure via reconstruction formulation with the chain-rule divergence evaluation, *J. Comput. Phys.* 232 (1) (2013) 7–13.
- [40] Z.J. Wang, Y. Liu, Extension of the spectral volume method to high-order boundary representation, *J. Comput. Phys.* 211 (2006) 154–178.
- [41] L. Krivodonova, M. Berger, High-order accurate implementation of solid wall boundary conditions in curved geometries, *J. Comput. Phys.* 211 (2006) 492–512.
- [42] G.S. Jiang, C.-W. Shu, Efficient implementation of weighted ENO schemes, *J. Comput. Phys.* 126 (1) (1996) 202–228.
- [43] Y.Z. Sun, Z.J. Wang, Evaluation of discontinuous Galerkin and spectral volume methods for scalar and system conservation laws on unstructured grids, *Int. J. Numer. Methods Fluids* 45 (8) (2004) 819–838.
- [44] M. Zhang, C.-W. Shu, An analysis and a comparison between the discontinuous Galerkin method and the spectral finite volume methods, *Comput. Fluids* 34 (2005) 581–592.
- [45] A. Meister, S. Ortelt, T. Sonar, M. Wirz, A comparison of the Discontinuous-Galerkin- and Spectral-Difference-Method on triangulations using PKD polynomials, *J. Comput. Phys.* 231 (23) (2012) 7722–7729.
- [46] C. Liang, C. Cox, M. Plesniak, A comparison of computational efficiencies of spectral difference method and correction procedure via reconstruction, *J. Comput. Phys.* 239 (2013) 138–146.
- [47] C.-W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, in: *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, Springer-Verlag, Berlin/New York, 1998, pp. 325–432.
- [48] C.-W. Shu, Total-variation-diminishing time discretization, *SIAM J. Sci. Stat. Comput.* 9 (1988) 1073–1084.
- [49] TauBench, <http://www.ipacs-benchmark.org/index.php?ts=download&unterseite=taubench>.
- [50] T.J. Chung, *Computational Fluid Dynamics*, Cambridge University Press, Cambridge, United Kingdom, 2002.
- [51] Y. Saad, M.H. Schultz, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, *SIAM J. Sci. Stat. Comput.* 7 (1986) 856–869.
- [52] Y.Z. Sun, Z.J. Wang, Y. Liu, Efficient implicit non-linear LU-SGS approach for compressible flow computation using high-order spectral difference method, *Commun. Comput. Phys.* 5 (2009) 760–778.