Adjoint-Based Error Estimation and hp-Adaptation for the High-Order CPR Method

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Adjoint-based adaptive methods have the capability of dynamically distributing computing power to the areas which are important for predicting the engineering output. In this paper, the correction procedure via reconstruction (CPR) method is extended to handle output-based adaptive hp-refinement. The continuous and discrete adjoint formulations for the CPR method are developed and the analysis of its dual consistent discretization is performed. Current method is applied to output-based adaptive simulations for 2D Euler equations. Several well-known inviscid flow cases are utilized to compare the effectiveness of the adjoint-based and the residual-based adaptation strategies.

I. Introduction

High order methods have the potential of delivering higher accuracy with less CPU time than lower order methods. However, for a compact scheme, the number of degrees of freedom (DOF) rises rapidly with the increased order of accuracy, which affects the prevalence of high-order methods in industry applications. Solution based adaptive methods has the capability of dynamically distributing computing power to the desired area, which can achieve required accuracy with minimal costs.1, 2, 3 For this reason, adaptive high-order methods have received considerable attention in the high-order CFD community.4, 5, 6, 7, 8, 9

The truncation error of a spatial discretization is determined by the mesh size and the order of the polynomial approximation. P-adaptations can be applied to smooth regions while h-adaptations are required for resolving compressible flow with shock waves or singularities.10, 11 The discretization of a compact scheme only depends on its local DOFs, which simplify the task of hp-adaptations involving complex geometries. Several compact high-order methods for unstructured meshes have been developed, such as the discontinuous Galerkin (DG) method,12, 10, 13 the spectral volume (SV) method14, 15 and the spectral difference (SD) method.16 Unlike the finite volume method that achieves high-order by expanding their reconstruction stencil, the above methods employ local DOFs to support high-order piecewise solution polynomials in each element, and the interaction between the local cell and its neighbors is represented by the common flux at the boundary. Recently, the flux reconstruction or the correction procedure via reconstruction (CPR) formulation was developed in 1D,17 and further extended in Ref. 18, 19, 20, 21, 22. It is a nodal differential formulation which can unite several well known high-order methods such as DG, SV and SD. The CPR formulation combines the compactness and high accuracy with the simplicity and efficiency of the finite difference method, and can be easily implemented for mixed unstructured meshes.

The effectiveness of adaptation methods highly depends on the accuracy of error estimations. There are at least three major types of adaptation criteria: gradient or feature based error estimates,23, 24, 25, 26 residual-based error estimates,27, 28, 29, 30, 31, 32 and adjoint-based error estimates.33, 34, 4, 5, 35, 36, 37, 6, 7, 8, 38 Heuristic feature-based criterion such as large gradient can not provide universal and robust error estimations.39, 5 The residual-based error indicator which is defined locally on each element has had some successes; however, it can lead to false refinements in convection-dominated flow. Adjoint-based error estimations are gaining a lot of research attention, which relates a specific functional output directly to the local residual by solving an additional adjoint problem. It can capture the propagation effects inherent in hyperbolic equations and has been shown very effective in driving a hp-adaptation procedure to obtain a very accurate prediction of the
functional output. Recently Fidkowski and P.L Roe developed a new error indicator based on the entropy variables to drive an hp-adaptation for inviscid and viscous flow. Entropy variables can be interpreted as the dual solution for the output of entropy balance in the whole domain. It can be obtained directly from the state variables without solving extra adjoint equations.\textsuperscript{40,41}

The adjoint solution is particularly important for error estimations and output-based adaptations. There are two approaches to obtain the adjoint solution for primal problems. We can solve the continuous adjoint equation which is a partial differential equation using any numerical method or directly solve the discrete adjoint equation derived from the discretized primal equation. It has been shown that the discrete adjoint solution leads to more accurate error estimations for the fine grid functional, while continuous adjoints gives better output estimation when the primal and adjoint solution are well resolved.\textsuperscript{42} However, the discrete adjoint solution should be consistent with the exact adjoint from the continuous adjoint equation. It is well known that the dual consistency can significantly impact the convergence of both the primal and adjoint approximations. There are several possible sources of dual inconsistency that can be introduced into a high-order discretization. A dual consistent discretization with semilinear forms such as the finite element and DG methods have been well examined for the Euler and Navier-Stokes equations.\textsuperscript{33,43,36,44} However, the analysis of dual consistency for differential-type methods has not been well investigated, which is one focus of the present paper.

The rest of the paper is organized as follows: In section 2 we briefly review the high-order CPR method. The continuous and discrete adjoint equations and the dual consistent discretization of the CPR method are studied in Section 3. Then section 4 describes adaptation strategies and procedures for hp-adaptations. Finally, Several numerical test cases of the inviscid flow are presented in Section 5.

II. Review of the CPR Method

For the sake of completeness, the CPR formulation is briefly reviewed. The CPR formulation was originally developed by Huynh in Ref. 17,45 under the name of flux reconstruction, and extended to simplex and hybrid elements by Wang & Gao in Ref. 18 under lifting collocation penalty. The authors later decided to employ the unified name CPR for the method. In Ref. 46, CPR was further extended to 3D hybrid meshes. The method is also described in two book chapters.\textsuperscript{47} CPR can be derived from a weighted residual method by transforming the integral formulation into a differential one. First, a hyperbolic conservation law can be written as

\begin{equation}
\frac{\partial Q}{\partial t} + \nabla \cdot \vec{F}(Q) = 0
\end{equation}

with proper initial and boundary conditions, where $Q$ is the state vector, and $\vec{F}(F,G)$ is the flux vector. Assume that the computational domain $\Omega$ is discretized into $N$ non-overlapping triangular elements $\{V_i\}_{i=1}^N$. Let $W$ be an arbitrary weighting function or test function. The weighted residual formulation of Eq. (1) on element $V_i$ can be expressed as

\begin{equation}
\int_{V_i} \left( \frac{\partial Q}{\partial t} + \nabla \cdot \vec{F}(Q) \right) W dV = 0.
\end{equation}

After applying integration by parts to the flux divergence, we can get

\begin{equation}
\int_{V_i} \frac{\partial Q}{\partial t} W dV + \int_{\partial V_i} W \vec{F}(Q) \cdot \vec{n} dS - \int_{V_i} \nabla W \cdot \vec{F}(Q) dV = 0.
\end{equation}

Let $Q_i$ be an approximate solution to the analytical solution $Q$ on $V_i$. On each element, the solution belongs to the space of polynomials of degree $k$ or less, i.e., $Q_i \in P^k(V_i)$ (or $P^k$ if there is no confusion) with no continuity requirement across element interfaces. Let the dimension of $P^k$ be $K = (k + 1)(k + 2)/2$. In addition, the numerical solution $Q_i$, for the moment, is required to satisfy Eq. (3) as

\begin{equation}
\int_{V_i} \frac{\partial Q_i}{\partial t} W dV + \int_{\partial V_i} W \vec{F}(Q_i) \cdot \vec{n} dS - \int_{V_i} \nabla W \cdot \vec{F}(Q_i) dV = 0.
\end{equation}

Obviously the surface integral is not properly defined because the numerical solution is discontinuous across element interfaces. Following the idea used in the Godunov method,\textsuperscript{48,49} the normal flux term in Eq. (4) is replaced with a common Riemann flux, e.g., in Ref. 50,51,52

\begin{equation}
F^n(Q_i) \equiv \vec{F}(Q_i) \cdot \vec{n} \approx F^n_{\text{com}}(Q_i, Q_{i+}, \vec{n}),
\end{equation}

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where $Q_{i+}$ denotes the solution outside the current element $V_i$. Instead of Eq. (4), the approximate solution is required to satisfy

$$\int_{V_i} \frac{\partial Q_i}{\partial t} W \, dV + \int_{\partial V_i} W F_{\text{com}} \cdot n \, dS - \int_{V_i} \nabla W \cdot \vec{F}(Q_i) \, dV = 0. \quad (6)$$

Applying integration by parts again to the last term of the above LHS, we obtain

$$\int_{V_i} \frac{\partial Q_i}{\partial t} W \, dV + \int_{V_i} W \nabla \cdot \vec{F}(Q_i) \, dV + \int_{\partial V_i} W [F_{\text{com}} - F^n(Q_i)] \cdot n \, dS = 0. \quad (7)$$

Here, the test space has the same dimension as the solution space, and is chosen in a manner to guarantee the existence and uniqueness of the numerical solution.

Note that the quantity $\nabla \cdot \vec{F}(Q_i)$ involves no influence from the data in the neighboring cells. The interaction between the current cell and its neighbors is represented by the above boundary integral, which is also called a penalty term, penalizing the normal flux differences.

The next step is critical in the elimination of the test function. The boundary integral above is cast as a lifting operator, which has the normal flux differences on the boundary as input and a member of $P^k(V_i)$ as output. Substituting Eq. (8) into Eq. (7), we obtain

$$\int_{V_i} \left( \frac{\partial Q_i}{\partial t} + \nabla \cdot \vec{F}(Q_i) + \delta_i \right) W \, dV = 0. \quad (9)$$

If the flux vector is a linear function of the state variable, then $\nabla \cdot \vec{F}(Q_i) \in P^k$. In this case, the terms inside the square bracket are all elements of $P^k$. Because the test space is selected to ensure a unique solution, Eq. (9) is equivalent to

$$\frac{\partial Q_i}{\partial t} + \nabla \cdot \vec{F}(Q_i) + \delta_i = 0. \quad (10)$$

For nonlinear conservation laws, $\nabla \cdot \vec{F}(Q_i)$ is usually not an element of $P^k$. As a result, Eq. (9) cannot be reduced to Eq. (10). In this case, the most obvious choice is to project $\nabla \cdot \vec{F}(Q_i)$ into $P^k$. Denote $\Pi(\nabla \cdot \vec{F}(Q_i))$ as a projection of $\nabla \cdot \vec{F}(Q_i)$ to $P^k$. One choice is

$$\int_{V_i} \Pi(\nabla \cdot \vec{F}(Q_i)) W \, dV = \int_{V_i} \nabla \cdot \vec{F}(Q_i) W \, dV. \quad (11)$$

Then Eq. (9) reduces to

$$\frac{\partial Q_i}{\partial t} + \Pi(\nabla \cdot \vec{F}(Q_i)) + \delta_i = 0. \quad (12)$$

With the introduction of the correction field $\delta_i$, and a projection of $\nabla \cdot \vec{F}(Q_i)$ for nonlinear conservation laws, we have reduced the weighted residual formulation to a differential formulation, which involves no explicit integrals. Note that for $\delta_i$ defined by Eq. (8), if $W \in P^k$, Eq. (12) is equivalent to the DG formulation, at least for linear conservation laws; if $W$ belongs to another space, the resulting $\delta_i$ is different. We obtain a formulation corresponding to a different method such as the SV method.

Next, let the DOFs be the solutions at a set of solution points (SPs) $\{\vec{r}_{i,j}\}$ ($j$ varies from 1 to $K$), as shown in Figure 1. Then Eq. (12) holds true at the SPs, i.e.,

$$\frac{\partial Q_{i,j}}{\partial t} + \Pi_j(\nabla \cdot \vec{F}(Q_i)) + \delta_{i,j} = 0, \quad (13)$$

where $\Pi_j(\nabla \cdot \vec{F}(Q_i))$ denotes the values of $\Pi(\nabla \cdot \vec{F}(Q_i))$ at SP $j$. The efficiency of the CPR approach hinges on how the correction field $\delta_i$ and the projection $\Pi(\nabla \cdot \vec{F}(Q_i))$ are computed. Two approaches can be used to
compute this divergence as detailed in Ref.\textsuperscript{18} To compute $\delta_i$, we define $k+1$ points named flux points (FPs) along each interface, where the normal flux differences are computed, as shown in Figure 1. We approximate (for nonlinear conservation laws) the normal flux difference $[F^n]$ with a degree $k$ interpolation polynomial along each interface

$$[F^n]_f \approx I_k[F^n]_f \equiv \sum_l [F^n]_{f,l} L_{f,l}^FP,$$

(14)

where $f$ is a face (or edge in 2D) index, and $l$ is the FP index, and $L_{f,l}^FP$ is the Lagrange interpolation polynomial based on the FPs in a local interface coordinate. For linear triangles with straight edges, once the solution points and flux points are chosen, the correction at the SPs can be written as

$$\delta_{i,j} = \frac{1}{|V_i|} \sum_f \sum_l \alpha_{j,f,l} [F^n]_{f,l} S_f,$$

(15)

where $\alpha_{j,f,l}$ are lifting constants independent of the solution variables, $S_f$ is the face area, $|V_i|$ is the volume of $V_i$. Note that the correction for each solution point, namely $\delta_{i,j}$, is a linear combination of all the normal flux differences on all the faces of the cell. Conversely, a normal flux difference at a flux point on a face, say $(f,l)$ results in a correction at all solution points $j$ of an amount $\alpha_{j,f,l} [F^n]_{f,l} S_f/|V_i|$.

III. Adjoint-Based Error Estimation

III.A. The Continuous Adjoint Equation

Adjoint-based error estimation can directly relate local residual error from the primal equation to the engineering output. So adjoint solutions are particularly important for the accuracy of error estimation. There are two approaches to obtain the adjoint for the primal problem. We can solve the continuous adjoint equation which is a partial differential equation using any numerical method or directly solve the discrete adjoint equation derived from the discretized primal equation. As for the primal problem, a numerical scheme is consistent if its discrete operator converges to the continuous operator, or the exact solution could satisfy the discrete numerical formulation. Similarly, dual-consistency is defined as the exact adjoint solution from the continuous adjoint equation should satisfy the discrete adjoint equation. In order to analyze the dual consistency of the CPR method, we need to derive the continuous adjoint equation and its boundary conditions first. Consider a primal differential equation as a conservation law

$$\mathcal{N}(Q) = \nabla \cdot \overline{F}(Q) = 0.$$

(16)

Given a scalar output $\mathcal{J}(Q)$ of interest, which may consist of surface($\partial \Omega$) and volume($\Omega$) integration in the general form

$$\mathcal{J}(Q) = \int_\Omega \mathcal{J}_\Omega(Q) d\Omega + \int_{\partial \Omega} \mathcal{J}_\tau(Q) ds,$$

(17)

we can define a Lagrangian of the output with the constraint of satisfying the primal equation $\mathcal{N}(Q) = 0$

$$\mathcal{L} = \mathcal{J}(Q) + \int_\Omega \psi^T \mathcal{N}(Q) d\Omega.$$

(18)
Here $\psi$ is the adjoint solution, furthermore, it also serves as a Lagrangian multiplier\textsuperscript{53,41} in the above equation. Let the Frechet linearization with respect to the arguments in the square bracket defined as

$$\mathcal{J}(Q)(\delta Q) = \mathcal{J}(Q + \delta Q) - \mathcal{J}(Q) = \delta \mathcal{J} = \frac{\partial \mathcal{J}}{\partial Q} \delta Q.$$  \hspace{1cm} (19)

After enforcing stationary of $\mathcal{L}$ to a permissible variations $\delta Q$, which is belong to the space of permissible state variations $\delta Q \in \mathcal{V}_{perm}$, Eq. (18) yields the linearized Lagrangian or the adjoint equation

$$\mathcal{L}(Q)(\delta Q) = \mathcal{J}(Q)(\delta Q) + \int_{\Omega} \psi^T \mathcal{N}[\mathcal{Q}(\delta Q)] \, d\Omega = 0 \quad \forall \delta Q \in \mathcal{V}_{perm}. \hspace{1cm} (20)$$

Plug the definition of Frechet linearization into the above equation, the adjoint Eq. (20) can be expressed as

$$\left(\frac{\partial \mathcal{J}}{\partial Q} + \int_{\Omega} \psi^T \frac{\partial \mathcal{N}(Q)}{\partial Q} \, d\Omega\right) \delta Q = 0. \hspace{1cm} (21)$$

### III.A.1. Error Estimation using Continuous Adjoint Approach

Substitute the definition of the output $\mathcal{J}$ and the primal differential equation $\mathcal{N}(Q)$ in the adjoint Eq. (21)

$$\left(\int_{\Omega} \frac{\partial \mathcal{J}}{\partial Q} \, d\Omega + \int_{\partial \Omega} \mathcal{J} \frac{\partial \mathcal{N}(Q)}{\partial Q} \, ds + \int_{\Omega} \psi^T \frac{\partial F_i}{\partial Q} \, d\Omega\right) \delta Q = 0,$$  \hspace{1cm} (22)

then performing integration by parts leads to

$$\left[\int_{\Omega} \frac{\partial \mathcal{J}}{\partial Q} \, d\Omega - \frac{\partial \psi^T}{\partial x_i} F_i \right] \delta Q + \int_{\partial \Omega} \frac{\partial \mathcal{J}}{\partial Q} \psi^T \frac{\partial F_i}{\partial Q} n_i \, ds \delta Q = 0. \hspace{1cm} (23)$$

Most of time, the output $\mathcal{J}$ only consists of boundary integration, which means $\mathcal{J}_{\Omega} = 0$, then the above equation yields the governing equation for the continuous adjoint

$$\frac{\partial F_i}{\partial Q} \frac{\partial \psi}{\partial x_i} = 0. \hspace{1cm} (24)$$

The boundary condition for the continuous adjoint equation is given as

$$\left[\int_{\partial \Omega} \left(\frac{\partial \mathcal{J}}{\partial Q} + \psi^T \frac{\partial F_i}{\partial n_i} \right) \, ds\right] \delta Q = 0 \quad \forall \delta Q \in \mathcal{V}_{perm}. \hspace{1cm} (25)$$

The continuous adjoint equation is a linear partial differential equation which can be solved using any numerical method. In order to perform the adjoint-based error estimation, first we define the output error as the difference between the output evaluated from the analytical solution $Q$ and the numerical solution $Q_h$

$$\delta \mathcal{J} = \mathcal{J}(Q_h) - \mathcal{J}(Q) \approx \mathcal{J}(Q)(\delta Q), \hspace{1cm} (26)$$

With Eq. (20), the output error can be derived as

$$\delta \mathcal{J} = \mathcal{J}(Q)(\delta Q) = -\int_{\Omega} \psi^T \mathcal{N}[\mathcal{Q}(\delta Q)] \, d\Omega, \hspace{1cm} (27)$$

where $\mathcal{N}(\mathcal{Q}(\delta Q)$ is the residual error for the primal problem induced by the primal discretization $\delta Q = Q - Q_h$

$$\delta \mathcal{N} = \mathcal{N}(Q_h) - \mathcal{N}(Q) = \mathcal{N}(Q_h) - \mathcal{N}(\mathcal{Q}(\delta Q)). \hspace{1cm} (28)$$

So we can express the output error in the form of the adjoint solution weighted the primal residual

$$\delta \mathcal{J} = -\int_{\Omega} \psi^T \mathcal{N}(Q_h) \, d\Omega. \hspace{1cm} (29)$$
III.A.2. Boundary Condition for the Adjoint of Euler Equations

Boundary conditions of the continuous adjoint $\psi$ is very important for the analysis of the dual consistency. From Eq. (25), the adjoint $\psi$ at the wall boundary should satisfy the following equation

$$-\psi^T \delta F n_i = \delta J_r \quad \forall \delta u \in V_{perm}, \quad (30)$$

where $n_i$ is the normal unit vector of the boundary surfaces. Very often, output for inviscid flow, such as drag coefficient or lift coefficient, is just a function of pressure. Therefore, the output perturbation in the above equation can be expressed as

$$\delta J_r = \delta p \cdot \vec{n} \cdot \vec{i}_d, \quad (31)$$

where $\vec{i}_d$ is a unit vector of the output of interest. For example, an unit vector $\vec{i}_D$ which is along the direction of the income flow should be used for an output of drag coefficient. The LHS of the boundary condition Eq. (30) only depends on the normal flux across the wall boundaries. In 2D case, it can be expressed as

$$F^n = (\begin{array}{c} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E + p) u \end{array} n_x + (\begin{array}{c} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E + p) v \end{array} n_y) \quad (32)$$

So the normal flux perturbation can be written as

$$\delta \mathcal{F}_i n_i = \delta F^n = \delta p (\begin{array}{c} 0 \\ n_x \\ n_y \\ 0 \end{array}). \quad (33)$$

From the Eq. (33) and Eq. (31), pressure perturbation $\delta p$ can be canceled and the boundary condition of the adjoint $\psi$ for the Euler equations is in the following form

$$-\psi^T (\begin{array}{c} 0 \\ n_x \\ n_y \\ 0 \end{array}) = \vec{n} \cdot \vec{i}_d. \quad (34)$$

III.B. The Discrete Adjoint Equation

As shown in the previous session, the continuous adjoint equation is a partial differential equation, which is derived directly from the linearized primal governing equation and the linearized functional outputs. Obviously, the numerical scheme for the continuous adjoint equation and the primal governing equation can be different. In the discrete adjoint approach, the adjoint equation is derived from the discretized governing equation. In this session, the discrete adjoint formulations for semilinear schemes and differential-type schemes are presented and their difference are compared.

III.B.1. The Variational Formulation for the Discrete Adjoint

Consider a general semilinear form arising from a weighted residual formula using the Galerkin method, which is finding a discretized solution $Q_h \in V_h$ satisfy

$$\mathcal{R}_h(Q_h, v_h) = 0 \quad \forall v_h \in V_h, \quad (35)$$

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where \( v_h \) is the testing function belongs to the same discrete solution space \( V_h \) and semilinear operator \( R_h(\cdot, \cdot) \) is linear in the second argument. Let’s rewrite the adjoint Eq. (20) derived in the previous section here for reference

\[
J[Q](\delta Q) = - \int_{\Omega} \psi^T N|Q|(\delta Q) \, d\Omega \quad \forall \delta Q \in V^{perm}.
\]

Replacing the solution perturbation \( \delta Q \) with the testing function \( v_h \) and with the help of the semilinear operator, the above equation can be changed to the semilinear form as

\[
J[Q](v_h) = - R_h(Q_h)(v_h, \psi_h) \quad \forall v_h \in V^{perm}.
\]  

(36)

Again from Eq. (29), the output error

\[
\delta J = - \int_{\Omega} \psi^T N(Q_h) \, d\Omega
\]

(37)

can be rewritten in the semilinear form too

\[
\delta J = - R_h(Q_h, \psi_h).
\]

(38)

### III.B.2. The Fully Discrete Formulation for the Discrete Adjoint

For the fully discrete formulation, we need to consider two discretization approximation levels. Here, \( H \) stands for a coarse level and \( h \) denotes a fine level. In practise, the coarse and fine level of approximations can be achieved using \( h \)-refinement or \( p \)-enrichment. The output error can be defined using those two different approximation levels

\[
\delta J_h = J_h(Q_H) - J_h(Q_h).
\]

(39)

Our target is to estimate the output \( J_h(Q_h) \) at fine level without solving the primal solution on the fine space. Using Taylor expansion, the output \( J_h(Q_h) \) and the residual \( R_h(Q_h) \) at the fine space can be expanded to a reconstructed fine solution from a coarse solution \( Q_H^h \)

\[
J_h(Q_h) = J_h(Q_H^h) + \frac{\partial J_h}{\partial Q_h} |_{Q_H^h} (Q_h - Q_H^h) + ...
\]

(40)

\[
R_h(Q_h) = R_h(Q_H^h) + \frac{\partial R_h}{\partial Q_h} |_{Q_H^h} (Q_h - Q_H^h) + ...
\]

where \( R_h(Q_h) = 0 \). The reconstructed solution \( Q_H^h = I_H^h Q_H \) can be obtained using the injection operator \( I_H^h \). After dropping high order terms and canceling \( (Q_h - Q_H^h) \) in the equations, the above equation can be written as

\[
\begin{align*}
J_h(Q_h) &\approx J_h(Q_H^h) - \frac{\partial J_h}{\partial Q_h} |_{Q_H^h} \frac{\partial R_h}{\partial Q_h} |_{Q_H^h} (Q_h - Q_H^h) \\
&\approx J_h(Q_H^h) + (\hat{\psi}_h | Q_H^h)^T R_h(Q_H^h)
\end{align*}
\]

(41)

from where we can define the equation for the discrete adjoint solution \( \hat{\psi}_h \) as

\[
-(\hat{\psi}_h)^T \frac{\partial R_h}{\partial Q_h} = \frac{\partial J_h}{\partial Q_h}.
\]

After transposing both sides of the above equation, we can express the fully discrete adjoint equation in the following form

\[
- \frac{\partial R_h}{\partial Q_h}^T \hat{\psi}_h = \frac{\partial J_h}{\partial Q_h}^T.
\]

(42)
III.B.3. The Fully Discrete Adjoint for Numerical Schemes in Differential Form

After choosing a proper basis for the weighted residual formula using the Galerkin method, Eq. (42) is equivalent to its variational formulation given in Eq. (36). Detailed derivations can be found in Ref. 43, 9. The fully discrete adjoint solution for numerical schemes in semilinear form is consistent with the continuous adjoint equation. However, this is not true for a numerical scheme in differential forms.

Let \( r(Q)_{i,j} \) denotes a pointwise residual of a differential scheme defined at each solution point \( j \) of cell \( i \)

\[
r(Q)_{i,j} = (\nabla \cdot \vec{F}(Q_i))_j. \tag{43}
\]

If the CPR method is used for discretizing the primal equation, the pointwise residual \( r_{i,j} \) can be expressed as

\[
r(Q)_{i,j} = \Pi(\nabla \cdot \vec{F}(Q_i))_j + \frac{1}{|V_i|} \sum_f \sum_l \alpha_{j,f,l} [\mathcal{F}^n]_{f,l} S_f. \tag{44}
\]

Substitute the pointwise residual \( r_{i,j} \) arising from a differential scheme, the fully discrete adjoint Eq. (42) can be written as

\[
- \sum_i \sum_j \frac{\partial r_{i,j}}{\partial Q_k} \hat{\psi}_{i,j} = \frac{\partial J}{\partial Q_k}, \tag{45}
\]

where \( k \) is index of total DOFs in the whole domain. However, this approach is not consistent with continuous adjoint equation. If we think the adjoint solution belongs to the same space of the primal solution and approximate the adjoint variable \( \psi \) of the cell \( i \) using the Lagrange basis \( L_j \)

\[
\psi_i = \sum_j L_j \hat{\psi}_{i,j}. \tag{46}
\]

With the above equation, directly discretizing the continuous adjoint Eq. (21) leads to

\[
- \int_{\Omega} \frac{\partial N(Q)}{\partial Q}^T \psi d\Omega = \frac{\partial J}{\partial Q}^T
\Rightarrow - \sum_i \sum_j \frac{\partial r_{i,j}}{\partial Q_k} \omega_j |J_{i,j}| \hat{\psi}_{i,j} = \frac{\partial J}{\partial Q_k}, \tag{47}
\]

where \( \omega_j \) and \( |J_{i,j}| \) are the quadrature weight and the element Jacobian at the solution point \( j \) of cell \( i \). Compared with Eq. (45), the following relation can be derived between the discrete adjoints \( \hat{\psi}_{i,j} \) and the continuous adjoint \( \hat{\psi}_{i,j} \)

\[
\hat{\psi}_{i,j} = \omega_j |J_{i,j}| \hat{\psi}_{i,j}. \tag{48}
\]

So the fully discrete adjoint formula for numerical schemes in differential forms is not consistent with the continuous adjoint equation. The only difference between them are the quadrature weights \( \omega \) and cell Jacobian \( |J| \) at each solution point. The discrete adjoint formula for differential schemes should be derived in integral form. In the next session, a discrete adjoint equation in integral form for the CPR method, which is dual consistent with the continuous adjoint, is presented and verified using numerical tests.

III.C. The Analysis of Dual-consistency for the CPR Method

For the CPR method, the discrete adjoint equation in integral forms is

\[
- \sum_i \sum_j \frac{\partial r_{i,j}}{\partial Q_k} \omega_j |J_{i,j}| \hat{\psi}_{i,j} = \frac{\partial J}{\partial Q_k}, \tag{49}
\]

which is directly derived from the continuous adjoint equation. Substitute the residual of CPR method

\[
r(Q) = \nabla \cdot \vec{F} + \delta. \tag{50}
\]
where $\delta$ is the correction field, in the above equation, we get
\[
\int_{\Omega} \psi^T \frac{\partial}{\partial Q} (-\nabla \cdot \bar{F} - \delta) \, d\Omega = \frac{\partial J}{\partial Q}.
\] (51)

For a dual consistent discretization, the analytic primal solution $Q$ and analytic dual solution $\psi$ should satisfy the discrete equation. The LHS of above equation can be expanded as
\[
LHS = -\int_{\Omega} \psi^T \nabla \cdot \frac{\partial \bar{F}}{\partial Q} \, d\Omega - \int_{\Omega} \psi^T \frac{\partial \delta}{\partial Q} \, d\Omega.
\] (52)

Assume the computational domain $\Omega$ is partitioned into a set of $N$ non-overlapping elements as
\[
\Omega = \bigcup_{k=1}^{N} \Omega_k, \quad k \in T_h,
\] (53)

where $T_h = \{k\}$ is the set of all elements $k$ in the mesh size $h$. The Eq. (52) can be written on each element as
\[
LHS = -\sum_k \int_{\Omega_k} \psi^T \nabla \cdot \frac{\partial \bar{F}}{\partial Q} \, d\Omega - \sum_k \int_{\Omega_k} \psi^T \frac{\partial \delta}{\partial Q} \, d\Omega.
\] (54)

Performing integration by parts twice, the above equation becomes
\[
LHS = \sum_k \int_{\Omega_k} \frac{\partial \bar{F}}{\partial Q} \nabla \psi^T \, d\Omega - \sum_k \int_{\partial \Omega_k} \psi^T \frac{\partial \bar{F}^n}{\partial Q} \, ds - \int_{\Gamma} \psi^T \frac{\partial \bar{F}^n}{\partial Q} \, ds - \sum_k \int_{\Omega_k} \psi^T \frac{\partial \delta}{\partial Q} \, d\Omega,
\] (55)

where $\Gamma$ denotes the outer boundaries of the whole computational domain and $\partial \Omega_k/T$ denotes the inner element’s boundaries.

For the CPR method, the key step to verify its dual consistency is to lower the multiplication of the adjoint solution $\psi$ and the correction field $\delta$ from the volume integration to the boundary integration. Since our correction field $\delta$ satisfies the follow equation for each cell $T_i$
\[
\int_{T_i} \phi \delta \, d\Omega = \int_{\partial T_i} \phi [\bar{F}^n] \, ds \quad \forall \phi \in \mathcal{V}_h.
\] (56)

In addition, we assume the adjoint $\psi$ belongs to the same space of $\mathcal{V}_h$, so we have
\[
\int_{T_i} \psi^T \delta \, d\Omega = \int_{\partial T_i} \psi^T [\bar{F}^n] \, ds \quad \forall \psi \in \mathcal{V}_h.
\] (57)

After lowering the correction field $\delta$ back to the surface integration, Eq. (55) can be expressed as
\[
LHS = \sum_k \int_{\Omega_k} \frac{\partial \bar{F}}{\partial Q} \cdot \nabla \psi^T \, d\Omega - \sum_k \int_{\partial \Omega_k/T} \psi^T \frac{\partial \bar{F}^n}{\partial Q} \, ds - \int_{\Gamma} \psi^T \frac{\partial \bar{F}^n}{\partial Q} \, ds - \sum_k \int_{\Omega_k} \psi^T \frac{\partial \delta}{\partial Q} \, d\Omega.\] (58)

As we can see, if the Lagrange polynomial(LP) approach is utilized to evaluating the flux divergence $\nabla \cdot \bar{F}$, the following equation can be satisfied exactly at the cell boundaries
\[
\int_{\partial \Omega_k} \psi^T \frac{\partial \bar{F}}{\partial Q} \, ds + \int_{\partial \Omega_k} \psi^T \frac{\partial \bar{F}^n}{\partial Q} \, ds = \int_{\partial \Omega_k} \psi^T \frac{\partial \bar{F}(Q_i, Q_r, \bar{n})}{\partial Q} \, ds.
\] (59)

However, it is not true for the chain-rule(CR) approach 18. With this flux conservation property, we can express Eq. (58) into the following form
\[
= \sum_k \int_{\Omega_k} \frac{\partial \bar{F}}{\partial Q} \cdot \nabla \psi^T \, d\Omega - \sum_k \int_{\partial \Omega_k/T} \psi^T \frac{\partial \bar{F}(Q_i, Q_r, \bar{n})}{\partial Q} \, ds - \int_{\Gamma} \psi^T \frac{\partial \bar{F}(Q_i, Q_{bc}(Q_i), \bar{n})}{\partial Q} \, ds.
\] (60)

For a dual consistent discretization, all of terms in Eq. (60) should be vanished. Let’s examine Eq. (60) term by term:
The interior volume term ($\Omega_k$)

The interior volume integration term satisfies the governing equation of the continuous adjoint (24) exactly, so it vanishes immediately.

$$\int_{\Omega_k} \frac{\partial \mathcal{F}}{\partial Q} \cdot \nabla \psi^T d\Omega = 0.$$ 

The interior boundary term ($\partial k/\Gamma$)

For each interior surface, the solution perturbation affects two common fluxes which are from its face-neighbor cells. For the common flux of current cell, the solution perturbation is denoted as $Q^+$, while the common flux belongs to the neighbor cells is denoted as $Q^-$. Furthermore, based on the conservation property of the common flux, we know that $\tilde{F}(Q_l, Q_r, \vec{n}_l) = -\tilde{F}(Q_l, Q_r, \vec{n}_r)$. So we can prove those interior boundary terms can be vanished after performing following steps

$$- \sum_k \int_{\partial k/\Gamma} \psi^T \frac{\partial \tilde{F}(Q_l, Q_r, \vec{n})}{\partial Q} d\Gamma = - \sum_k \int_{\partial k/\Gamma} \psi^T \left( \frac{\partial \tilde{F}(Q_l, Q_r, \vec{n})}{\partial Q_l} + \frac{\partial \tilde{F}(Q_l, Q_r, \vec{n})}{\partial Q_l} \right) d\Gamma
= - \sum_k \int_{\partial k/\Gamma} \left( \psi^T_l - \psi^T_r \right) \frac{\partial \tilde{F}(Q_l, Q_r, \vec{n}_l)}{\partial Q_l} d\Gamma
= 0 \text{ when } \psi_l = \psi_r \rightarrow \psi.$$

The outer boundary term ($\Gamma$)

The lessons we learn from the dual consistent semilinear methods$^{33,43}$ such as the requirements of the special numerical flux near the wall boundaries and the special output definitions still play an important role for the differential-type schemes. Note that the domain boundary term from the LHS should be balanced to the RHS of the discrete adjoint equation in the following form

$$- \int_{\Gamma} \psi^T \frac{\partial \tilde{F}(Q_l, Q_{bc}(Q_l), \vec{n})}{\partial Q} d\Gamma = \int_{\Gamma} \frac{\partial j_{\Gamma}}{\partial Q} d\Gamma. \quad (61)$$

After performing the chain rule on the common flux $\tilde{F}(Q_l, Q_{bc}(Q_l), \vec{n})$ and substitute the output $j_{\Gamma} = p\vec{n} \cdot \vec{i}_{dir}

$$- \int_{\Gamma} \psi^T \left( \frac{\partial \tilde{F}(Q_l, Q_{bc}(Q_l), \vec{n})}{\partial Q_l} + \frac{\partial \tilde{F}(Q_l, Q_{bc}(Q_l), \vec{n})}{\partial Q_{bc}} \frac{\partial Q_{bc}}{\partial Q_l} \right) d\Gamma = \int_{\Gamma} \frac{\partial p}{\partial Q} \vec{n} \cdot \vec{i}_{dir} d\Gamma. \quad (62)$$

Note that the continuous adjoint boundary conditions defined in Eq. (34) as

$$-\psi^T \begin{pmatrix} 0 \\ n_x \\ n_y \\ 0 \end{pmatrix} = \vec{n} \cdot \vec{i}_{dir}.$$

Therefore, in order to satisfy the above equation, the LHS should only have one term or the common flux at the boundary should only depend on $Q_{bc}$.

$$- \int_{\Gamma} \psi^T \frac{\partial \tilde{F}(Q_{bc}(Q_l), \vec{n})}{\partial Q_{bc}} \frac{\partial Q_{bc}}{\partial Q_l} d\Gamma = \int_{\Gamma} \frac{\partial p}{\partial Q} \vec{n} \cdot \vec{i}_{dir} d\Gamma. \quad (63)$$
This is only possible if
\[ \bar{F}(Q_{bc}(Q_l), \vec{n}) = \vec{n} \cdot \bar{F}(Q_{bc}(Q_l)) \]
\[ = p \begin{pmatrix} 0 \\ n_x \\ n_y \\ 0 \end{pmatrix}, \]  
\[ (64) \]

Finally, with the above equation, Eq. (62) becomes
\[ -\int_{\Gamma} \frac{\partial p}{\partial Q_l} \psi^T \begin{pmatrix} 0 \\ n_x \\ n_y \\ 0 \end{pmatrix} \, ds = \int_{\Gamma} \frac{\partial p}{\partial Q} \vec{n} \cdot \vec{i}_{ds} \, ds, \]
\[ (65) \]

which indicates the balance of the LHS and the RHS as in the form of the boundary conditions for the continuous adjoint equation
\[ -\psi^T \begin{pmatrix} 0 \\ n_x \\ n_y \\ 0 \end{pmatrix} = \vec{n} \cdot \vec{i}_d. \]  
\[ (66) \]

Now all of the terms in Eq. (60) vanished with the analytical primal and adjoint solutions, which means the CPR method with the LP approach and the special numerical flux at the wall boundaries is a dual consistent discretization formulation. However, a special care need to carry out for the solution points configuration, which can affect the quadrature accuracy for the integration in the discrete adjoint equation. In this paper, the Gaussian quadrature points for simplex given in Ref. 54 are used for the location of the solution points and the 1D Gaussian quadrature points are used for the coordinate of the flux points.

III.D. Numerical Verifications of the Dual Consistent Discretization of the CPR Method

Inviscid flow over a NACA 0012 airfoil is utilized to demonstrate the smoothness of the discrete adjoint solution from the dual consistent CPR method. This test case is used in Ref. 9. The inflow condition is set to be \( M = 0.4 \) with an angle of attack of 5°. The 3rd order CPR formulation using Gaussian quadrature points as solution points and flux points is used to ensure the integration accuracy for solving the discrete adjoint equation in integral forms. The dual solution from the fully discrete adjoint formulation of the CPR method is shown in Figure 2(a), which is not dual consistent and has an irregular mosaic like distributions in every cell. Figure 2(b) shows the adjoint solution from the discrete adjoint equation in integral forms. The adjoint solution looks much more smooth than the fully discrete adjoint. Furthermore the square root singularity of adjoint solution with respect to distance from the stagnation streamline introduced in Ref. 33 are observed near the leading edge.

The problem C1.3(a) of the 1st international workshop on high-order methods is used to further assess the accuracy of the adjoint-based error estimation with the CPR method. This test case involves subsonic flow over a NACA 0012 airfoil with a free-stream Mach number of \( M = 0.5 \) and the angle of attack, \( \alpha = 2^\circ \). The output of interest is chosen as the lift of the airfoil. The error in the functional \( J_H(Q_H) - J_h(Q_h) \) is computed using p-enrichment from \( p = 1 \) to \( p = 2 \) and the effectivity of the error estimation is defined as
\[ \eta_H = \frac{-(\psi_h)^T R_h(Q_h^H)}{J_H(Q_H) - J_h(Q_h)} \]  
\[ (67) \]

Table 1 on the following page shows the results with 4 levels of uniformly refined meshes from the high-order workshop. Note that the error of the initial lift estimation on the very coarse meshes are kind of large; however, the effectivity index \( \eta_H \) approaches unity as the mesh refined.
Table 1. Adjoint-based Error Estimation for the lift of a Subsonic NACA 0012 Airfoil at $M_{\infty} = 0.5, \alpha = 2^\circ$

<table>
<thead>
<tr>
<th>Cells</th>
<th>$J_h(Q_H) - J_h(Q_h)$</th>
<th>$- (\psi_h)^T R_h(Q_H)$</th>
<th>$\eta_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>280</td>
<td>-5.859e-3</td>
<td>-1.103e-2</td>
<td>1.88</td>
</tr>
<tr>
<td>1120</td>
<td>-2.638e-3</td>
<td>-4.002e-3</td>
<td>1.52</td>
</tr>
<tr>
<td>4480</td>
<td>-8.736e-4</td>
<td>-9.995e-4</td>
<td>1.14</td>
</tr>
<tr>
<td>17920</td>
<td>-1.933e-4</td>
<td>-1.988e-4</td>
<td>1.03</td>
</tr>
</tbody>
</table>

In this test case, subsonic inviscid flow over a Gaussian-shaped bump is used to demonstrate the implications of the boundary flux for the discrete adjoint solutions near the walls. Again, 3rd order CPR formulation using Gaussian quadrature points as the solution points and the flux points is employed. This test case gets rid of the influence from geometry singularities and stagnation points, which is first used in Ref. 53. The channel has an height of 0.8 unit and a length of 3 unit. The bump geometry is defined as

$$y = 0.0625e^{-25x^2}$$

and the output is defined as a weighted lift on bump surface

$$J = \int_{bump} p(x, y) n_y e^{-50x^2} \, ds.$$ 

The characteristic boundary conditions are used at both the inlet and outlet. The inflow Mach number is set to be $M = 0.5$. To impose the boundary conditions on the no-slip walls in a dual-consistent manner, the common flux defined in Ref. 33, 43, 53 is used. As shown in Figure 3, the resulting discrete adjoint using the inconsistent boundary conditions have significant irregularity near the wall boundary, whereas the adjoints using the dual-consistent common flux is quite smooth in the whole domain.

IV. Adjoint-based hp-Adaptations

IV.A. Adjoint-based and Multi-p Residual-based Error Indicators

Adjoint-based error estimation relates a specific functional output directly to the local residuals by the adjoint solution, which can capture the propagation effects inherent in the hyperbolic equations. Therefore, the adjoint-based error estimates can form an effective error indicator to drive an adaptive refinement toward any engineering output. From the Eq. (39) and Eq. (70), we can estimate the output error as

$$\delta J_h(Q_h) = J_h(Q_H) - J_h(Q_h)$$

$$\approx - (\psi_h)^T R_h(Q_H),$$

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Figure 3. The x-momentum component of the weighted lift adjoint for a Gaussian-shaped bump where the fine solution $Q_h$ is approximated by prolongating from the low order to the high order discretization through

$$Q^H_h = I^H_h Q_H.$$  \hfill (71)

Then the adjoint-based local error indicator can be defined as

$$\eta = |(\psi_h)^T R_h(Q^H_h)|.$$  \hfill (72)

A so-called multi-p residual-based local error indicator can be obtained by evaluating the discretization residuals on the prolongated solution $Q^H_h$ only:

$$\eta = |R_h(Q^H_h)|.$$  \hfill (73)

Essentially the multi-p residual-based error estimator is a unweighted version of the adjoint-based error indicator with uniform adjoint $\psi = 1$ everywhere.

IV.B. The hp-Adaptation and Non-conforming interfaces

The error indicators defined above can be used to drive a fixed-fraction isotropic hp-adaptation. In this approach, a certain fraction $f$ of the current elements with the largest local error indicators are marked for either $h$-refinement or $p$-enrichment. It is well known that $h$-refinement should be applied to the discontinuities and $p$-enrichment is appropriate in smooth flow regions. The jump indicator defined as

$$\phi_k = \frac{1}{|\partial \Omega_k|} \int_{\partial \Omega_k} \left[ \frac{[q]}{\{q\}} \cdot \mathbf{n} \right] |d\mathbf{s}|,$$  \hfill (74)

which is introduced in Ref. 55 is used to examine the smoothness of the primal solution and determine which type of adaptation should be performed. Here, the pressure is used as the jump indicator variable $q$ throughout this paper. The surface average operator $\{ \cdot \}$ and the surface jump operator $[\cdot]$ are defined as

$$\{q\} = \frac{1}{2} (q^+ + q^-)$$  \hfill (75)

$$[q] = \mathbf{n} (q^+ - q^-),$$

where $(\cdot)^+$ and $(\cdot)^-$ notations refer to the elements on each side of the edge. The criterion between whether to perform $h$-refinement or $p$-enrichment is defined as

$$\begin{cases} 
\phi_k > \frac{1}{K}, & h - refinement \\
\phi_k < \frac{1}{K}, & p - enrichment 
\end{cases}$$  \hfill (76)
where $\mathcal{K} = 25$ suggested in Ref. 56 and Ref. 57 is used. When a cell is marked as h-refinement, the cell is divided into four new sub-elements and their solution order are set to be the same as their parents. As shown in Figure 4, non-conforming interfaces between cells with different h or p adaptation levels are created during the simulations. In order to maintain the smoothness of the solution, the order of cells with hanging nodes after the h-adaptation will be restricted to the same value. In addition, at most one level of order difference is allowed for p-enrichment. Special treatment is required when computing the common numerical flux on those non-conforming interfaces. Basically, a $L_2$ projection approach is used to preserve conservation and maintain accuracy. Detailed procedures can be found in Ref. 31.

IV.C. The hp-Adaptation Procedure

For an hp-adaptation of a steady problem, the adjoint-based adaptation and the multi-p residual-based adaptation almost share the same procedure. Following are the main steps:

1. Solve the flow field until a convergence criterion is reached.
2. Follow the procedure in previous sections to compute the error indicator $\eta$ for each cell.
3. Mark cells with the largest error indicator with the fixed fraction $f$.
4. Perform the h-refinements on the marked cells with large jumps and apply the p-enrichment on the rest of them.
5. Update metric terms and connectivities of the marked cells and their neighbors.
6. Inject the current solution into the new mesh and the new solution space.
7. Repeat step 1-6.

V. Numerical Results

V.A. Subsonic Flow over a Gaussian-shaped Bump

The test case of inviscid flow over a Gaussian-shaped bump is used again to assess the effectiveness of h and p-adaptation. The geometry of the bump surface and boundary conditions are chosen to be the same as the previous section. Only the residual-based error estimation is used to drive the adaptation. The simulation starts from uniform 2$^{nd}$ order scheme, and then performs five loops of adaptations. A fixed fraction of adaptation on each level is chosen to be $f = 0.1$. As shown in Figure 5 on the next page, both of h and p-adaptation are mainly performed around the bump surface. The result shows that the residual-based indicator clearly identify the error source, which is the bump surface in this case. After performing 5 adaptation iterations, the non-conforming surfaces with h and p hanging nodes do not pollute the solution smoothness. Compared with the result of uniform 2$^{nd}$ order scheme, the flow field at the final adaptation
level is much more smooth especially in the adapted area. $L_2$ entropy error over the whole domain is used as a measure of accuracy, which is defined as

$$\| \text{Error} \|_{L_2} = \sqrt{\frac{\int_{\Omega} \left( \frac{2p^*/p_\infty - p_\infty}{p_\infty} \right)^2 \, dV}{\int_{\Omega} dV}}. \quad (77)$$

For this simple case, the bump surface is quite smooth, which indicates a p-adaptation is preferred. Figure 6 on the following page shows the $L_2$ entropy error versus degrees of freedom with the residual-based h- and p-adaptation. Results of uniform h and p refinements are given for comparison too. Both of the h-adaptation and p-adaptation demonstrate better performance than their corresponding uniform refinements. As expected, p-adaptation is more effective than the h-adaptation in terms of the entropy error per degrees of freedom for this particular smooth problem. Furthermore, the uniform p refinement outperform the uniform h refinement, which indicates the advantage of high order discretizations in smooth regions.

![Figure 5. The residual-based h and p-adaptation for subsonic flow over a Gaussian-shaped bump](image)

(a) Mach number contours at the initial mesh with $P = 1$

(b) Mach number contours after 5 level p-adaptations

(c) The final mesh after 5 level h-adaptations

(d) The order distributions after 5 level p-adaptations

Figure 5. The residual-based h and p-adaptation for subsonic flow over a Gaussian-shaped bump
V.B. Subsonic Flow over a NACA 0012 Airfoil

The next test case involves subsonic flow over a NACA 0012 airfoil with a free-stream Mach number of $M = 0.5$ and the angle of attack, $\alpha = 2^\circ$. This test case is the problem C1.3(a) of the 1st international workshop on high-order methods and the 'truth' lift coefficient of 0.2865 and drag coefficient of $2.4219e^{-6}$ are chosen from the workshop results. The initial simplex mesh is shown in Figure 7(a). Again the adaptation starts from a 2nd order scheme, undergoes five levels of h-adaptation and p-adaptation separately with fraction $f=0.1$. Both of the adjoint-based adaptation and multi-p residual-based adaptation are investigated. The drag coefficient and lift coefficient are considered as the output of interest in this case. Compared with the p1 solution on the initial mesh illustrated in Figure 7, the Mach number contours after 5 adaptation iterations are particular smooth.

The final hp-adapted meshes and order distributions of each strategy are shown in Figure 10. The region near the trailing edge and the leading edge is adapted consistently for all types of error indicators. However, only the adjoint-based adaptation target the stagnation streamline to perform mesh refinements and order increment. This unique property of the output based adaptation is due to the singularity of the adjoint solution along the stagnation streamlines. Another different refinement regions between the output-based adaptation and the residual-based adaptation are the upper and lower surfaces of the airfoil. Intuitively, those areas are important for accurate prediction of the outputs related to the total force. Both of the adjoint-based adaptations with lift and drag coefficient as output correctly target those areas, whereas the residual-based adaptations leaves those region relatively coarse. In general, adjoint-based adaptations demonstrate its capability of capturing the propagation effects inherent in Euler equations.

Figure 8 compares the lift coefficient error, drag coefficient error and entropy error of the different adaptation strategies with the result from the uniform h and p refinements. It is clearly to see that the adaptive methods could produce much more efficient error reductions in term of the DOFs. The adjoint-based adaptations with lift coefficient and drag coefficient perform similarly and are better than the residual-based adaptations. For this case, the p-adaptations performs better than the h-adaptations due to the smoothness of the solution.

Further comparisons of the h-adaptations with different approximation orders are shown in Figure 9. It is well known that the solution discontinuities can affect the optimal order of accuracy of the high order methods. Since there is a geometry singularity point at the trailing edge of the airfoil, the uniform mesh refinements can not achieve their expected order. The current results show that the h-adaptations successively refine the mesh around the trailing edge; therefore it could reduce the effect of this geometry singularity and reveal the potential accuracy from the high order CPR method. As shown in the Figure 9, the accuracy per DOFs from the h-adaptations with $P=2$ polynomials are much better than the h-adaptations with $P=1$ polynomials, which demonstrate the benefits of the high-order approximations.

V.C. Transonic Flow over a NACA 0012 Airfoil

The last test case is a NACA 0012 airfoil in inviscid transonic flow with an inflow Mach number $M = 0.8$ and the angle of attack, $\alpha = 1.25^\circ$. Again, this test is the problem C1.3(b) of the 1st international workshop on high-order methods and the 'truth' lift coefficient of 0.35169 and drag coefficient of 0.022628 are chosen.
(a) The initial mesh

(b) P=1 on the initial mesh

(c) 5 level p-adaptations using the lift adjoint

(d) 5 level h-adaptations using the lift adjoint

Figure 7. Adjoint-based and residual-based h- and p-adaptation for a NACA 0012 airfoil at $M_0 = 0.5$, $\alpha = 2^\circ$. 
from the workshop results. The structure of the solution to this transonic problem includes shockwaves on the upper and lower surfaces of the airfoil. The mesh shown in Figure 11(a) is used as the initial grid. Both of adjoint-based and residual-based hp-adaptations are tested. All of the calculations start with an uniform solution order of p=0 and consistently refined using previously described adaptation procedure. H-adaptations are only performed in the solution regions which are not smooth. The jump indicator given in Eq. (74) is used to identify non-smooth regions. Once a cell with large jumps is marked, the order of its solution polynomial is reduced to p=0. This approach can be treated as a spacial limiter; therefore there is no need to use artificial viscosity to stabilize the solution in the presence of shocks. So we don’t need to worry about any pollution to the adjoint solution by the artificial viscosities.

The final mesh and order distributions after 5 adaptation iterations are shown in Figure 11. The result shows that the shockwaves near the upper and lower surfaces are correctly identified and refined repetitively using h-adaption. P-adaptation is mainly applied to the leading edge, trailing edge and stagnation streamlines for the adjoint-based approaches. Mach number contours with the lift output and the residual-based approach at the final adaptation level are plotted in Figure 11(e) and Figure 11(f). Figure 12 shows the results driven by different adaptation indicators. For this particular transonic case, the adjoint-based adaptation with lift and drag as the outputs performs similarly and are much better than the residual-based adaptations.

VI. Conclusions

In this paper, the FR/CPR method is extended to handle output-based adaptive hp-refinement. The continuous and discrete adjoint formulations for the CPR method are developed and the analysis of its dual consistency is performed. The smoothness of discrete adjoint solutions from the adjoint consistent CPR formulation is demonstrated with numerical experiments. Several well-known two-dimensional inviscid flow cases are utilized to compare the effectiveness of adjoint-based and residual-based hp-adaptations. The refinement driven by the adjoint-based error indicator can directly target the error source to the engineering output. Results show significant savings of degrees of freedom when compared to the uniform h or p
Figure 9. The results of h-adaptation with P=1 and P=2 for a NACA 0012 airfoil at $M_0 = 0.5$, $\alpha = 2^\circ$.

refinement. The present hp-adaptation procedure was also demonstrated with success for a transonic flow over a NACA 0012 airfoil. It was shown that the adjoint-based hp-adaptation with the CPR method can effectively resolve important flow features which are related to the output of interest and has considerable potential in complex flow problems.

References


Figure 10. 5 level adaptations for a NACA 0012 airfoil with $M_{\infty} = 0.5$, $\alpha = 2^\circ$.
Figure 11. Hp-adaptation for a NACA 0012 airfoil at $M_0 = 0.8$, $\alpha = 1.25^\circ$
Figure 12. Hp-adaptation for a NACA 0012 airfoil at $M_0 = 0.8$, $\alpha = 1.25^\circ$.


