Recent Development On The Conservation Property of Chimera

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Abstract
An estimate on the conservation error due to the non-conservative data interpolation scheme for overset grids is given in this paper. It is shown that the conservation error is a first-order term if second-order conservative schemes are employed for the Chimera grids, and if discontinuities are located away from overlapped grid interfaces. Therefore, in the limit of global grid refinement, valid numerical solutions should be obtained with a data interpolation scheme. In one demonstration case, the conservation error in the original Chimera scheme was shown to affect flow even without discontinuities on coarse to medium grids. The conservative Chimera scheme was shown to give significantly better solutions than the original Chimera scheme on these grids with other factors being the same.

Introduction
The issue of conservation was brought up when the Chimera scheme was first proposed\(^1,2\). During the last one and half decades, Chimera grid methods have undergone significant development and have been applied to solve very complex steady and unsteady flow problems\(^3-10\). There are two opposing views on the issue of conservation regarding Chimera in the CFD community. One side claims that conservation is a non-issue. As long as the computational mesh is fine enough, a physically correct numerical solution should always be obtained. On the
contrary, the other side maintains that conservation is one fundamental requirement which should be satisfied by any numerical scheme. One should not trust the solution obtained with a non-conservative numerical procedure.

Although it is generally recognized that a non-conservative Chimera scheme may have difficulty in handling discontinuities, there is no consensus view on whether or not conservation is important for flows without discontinuities, i.e., smooth flows. Many CFD practitioners, therefore, drew their conclusions based on their own experiences with the Chimera grid method. For example, Davis\textsuperscript{11} reported that mass conservation must be guaranteed to correctly predict the thrust of a rocket engine. On the other hand, in Reference 12, Meakin showed that the formal order of accuracy was maintained with a non-conservative Chimera approach. It seems that the debate on the issue of conservation of Chimera is still on-going.

In this paper, an estimate on the conservation error due to the non-conservative data interpolation scheme is presented. The estimate shows that the conservation error is a first-order term if second-order conservative schemes are used in the Chimera grids, and the error should diminish with global grid refinement for smooth flows. On the other hand, if a shock wave coincides with a overlapped interface boundary, the computation may diverge due to large local errors around the shock wave. It was also demonstrated in one test case, for the first time, that a conservative Chimera scheme gave significantly better computational results than the original Chimera scheme on coarse to medium grids for smooth flows with other factors being the same. On a sufficiently fine mesh, the conservative Chimera and non-conservative Chimera schemes converged to the same solution. The authors wish to put an end to the debate on the issue of conservation regarding
An Estimate of Conservation Error in the Original Chimera Scheme

Consider the following one-dimensional conservation law:

\[
\frac{\partial u(x, t)}{\partial t} + \frac{\partial F(u)}{\partial x} = 0
\]  

(1)

with initial condition

\[
u(x, 0) = u_0(x)
\]  

(2)

where \( t > 0, x \in (-\infty, \infty) \), \( F(u) \) is bounded provided \( u \) is bounded. The space is partitioned into two overlapping subdomains \( A \) and \( B \) as shown in Figure 1. The overlapped region of \( A \) and \( B \) is \( O \), which is assumed fixed. Cells of \( A \) are indexed with \( i \) and the maximum cell index is \( I \). Similarly cells of \( B \) are indexed with \( j \) and the minimum index is \( 1 \). Fluxes are calculated at cell interfaces. Domain \( A \) and \( B \) each have a ghost cell with index \( I + 1 \) and 0, respectively, whose flow variables are interpolated from the other grid. If conservative schemes are used in both \( A \) and \( B \), i.e.,

\[
\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} \Delta x_i + \left( \tilde{F}_{i+\frac{1}{2}} - \tilde{F}_{i-\frac{1}{2}} \right) \right) = 0 \quad i \in A
\]  

(3)

\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{\Delta t} \Delta x_j + \left( \tilde{F}_{j+\frac{1}{2}} - \tilde{F}_{j-\frac{1}{2}} \right) \right) = 0 \quad j \in B
\]  

(4)

where \( u_{j}^{n} \) is the cell-averaged numerical solution of cell \( j \) at time \( n\Delta t \), \( \Delta t \) is the time step, and \( \tilde{F}_{i+\frac{1}{2}} \) is the numerical flux at face \( i+1/2 \). Summarizing Equations (3) and (4) over domains \( A \) and \( B \), we obtain
Note that indices $I + \frac{1}{2}$ and $\frac{1}{2}$ represent the boundaries of A and B, respectively. It is obvious that a conservative scheme for the entire computational domain must satisfy

$$\sum_{i \in A} \frac{\Delta u_i}{\Delta t} \Delta x_i + \left( \tilde{F}_{I+\frac{1}{2}} - \tilde{F}_{-\infty} \right) = 0$$ \hspace{1cm} (5)$$

$$\sum_{j \in B} \frac{\Delta u_j}{\Delta t} \Delta x_j + \left( \tilde{F}_{-\infty} - \tilde{F}_{j/2} \right) = 0$$ \hspace{1cm} (6)$$

Combining Equations (5), (6), and (7), then

$$\sum_{i \in A} \frac{\Delta u_i}{\Delta t} \Delta x_i + \sum_{j \in B} \frac{\Delta u_j}{\Delta t} \Delta x_j + \sum_{k \in O} \frac{\Delta u_k}{\Delta t} \Delta x_k + \tilde{F}_{-\infty} - \tilde{F}_{\infty} = 0$$ \hspace{1cm} (7)$$

This is the same condition also given in Reference 13, and is generally very difficult to satisfy unless overlapped regions are transformed into patched regions. In this paper, only steady-state problems are addressed and it is further assumed that a steady-state solution can always be obtained. Therefore, the criteria of conservation is simplified to

$$\tilde{F}_{I+\frac{1}{2}} - \tilde{F}_{I-\frac{1}{2}} = 0$$ \hspace{1cm} (9)$$

With a data interpolation scheme to exchange information between overlapped grids, Equation (9) is usually not satisfied. The conservation error is thus defined as

$$E = \left| \tilde{F}_{I+\frac{1}{2}} - \tilde{F}_{I-\frac{1}{2}} \right|$$ \hspace{1cm} (10)$$

Note that at steady-state
\[ \tilde{F}_{i+\frac{1}{2}} = \tilde{F}_{i-\frac{1}{2}} = \tilde{F}_{i+\frac{1}{2}} \quad i \in A \]  

and

\[ \tilde{F}_{j+\frac{1}{2}} = \tilde{F}_{j-\frac{1}{2}} = \tilde{F}_{j+\frac{1}{2}} \quad j \in B \]

i.e., the fluxes must be a constant on each grid. In the original Chimera scheme, flow variables at the ghost cells are interpolated from the other grids, i.e.,

\[ u_{I+1} = \alpha u_m + (1 - \alpha)u_{m+1}, \quad 0 \leq \alpha \leq I \]  

(13)

\[ u_o = \beta u_{I-k} + (1 - \beta)u_{I-k+1}, \quad 0 \leq \beta \leq I \]  

(14)

where the center of cell I+1 of A is bounded by the centers of cell m and cell m+1 of B, and the center of cell 0 of B is bounded by the centers of I-k and cell I-k+1 of A. Since the size of the overlapped region O is fixed, therefore

\[ k\Delta x_i = m\Delta x_j = L + \vartheta(\Delta x) \]  

(15)

where L is the length of region O. For a second order scheme, the following expressions hold

\[ \tilde{F}_{I+\frac{1}{2}} = \frac{1}{2}[F(u_I) + F(u_{I+\frac{1}{2}})] + \vartheta(\Delta x_i^2) \]  

(16)

\[ \tilde{F}_{\frac{1}{2}} = \frac{1}{2}[F(u_o) + F(u_I)] + \vartheta(\Delta x_j^2) \]  

(17)

where F represents the analytical flux function. Considering Equations (13) and (14), we can show that the conservations error is
\[ E = \left| \tilde{F}_{i + \frac{1}{2}} - \tilde{F}_{i - \frac{1}{2}} \right| = \]
\[ \frac{I}{2} \left[ F(u_i) - F(\beta u_{i-k} + (1 - \beta)u_{i-k+1}) \right] \]
\[ - \frac{I}{2} \left[ F(u_i) - F(\alpha u_m + (1 - \alpha)u_{m+1}) \right] + \vartheta(\Delta x^2) \]
\[ \leq \frac{I}{2} \left[ F(u_i) - F(\beta u_{i-k} + (1 - \beta)u_{i-k+1}) \right] \]
\[ + \frac{I}{2} \left[ F(u_i) - F(\alpha u_m + (1 - \alpha)u_{m+1}) \right] + \vartheta(\Delta x^2) \]

Since

\[ \tilde{F}_{i + \frac{1}{2}} = \frac{1}{2} \left[ F(u_i) + F(u_{i+1}) \right] + \vartheta(\Delta x^2) \]

\[ \tilde{F}_{i - \frac{1}{2}} = \frac{1}{2} \left[ F(u_{i-1}) + F(u_i) \right] + \vartheta(\Delta x^2) \]

and

\[ \tilde{F}_{i + \frac{1}{2}} = \tilde{F}_{i - \frac{1}{2}} \]

we therefore have:

\[ F(u_{i+1}) = F(u_{i-1}) + \vartheta(\Delta x^2). \]

Without loss of generality, assume that \( k \) is an even number. Hence

\[ F(u_i) - F(u_{i-k}) = \frac{k}{2} \vartheta(\Delta x^2) = \frac{k \Delta x}{2} \vartheta(\Delta x) = \vartheta(\Delta x) \]

In (23) the condition given in (15) is used. Therefore:

\[ F(u_i) = F(u_{i-k}) + \vartheta(\Delta x) \]
Applying Taylor expansion, then

\[ F(u_{I-j}) = F(\beta u_{I-j} + (1 - \beta)u_{I-j+1}) \]
\[ - \frac{\partial F}{\partial u} (I - \beta)(u_{I-j+1} - u_{I-j}) + \tilde{\theta}(\Delta x^2) \]  

(25)

From Equations (24) and (25), it is then obvious that

\[ F(u_I) - F(\beta u_{I-j} + (1 - \beta)u_{I-j+1}) \]
\[ = - \frac{\partial F}{\partial u} (I - \beta)(u_{I-j+1} - u_{I-j}) + \tilde{\theta}(\Delta x) \]  

(26)

If \( \frac{\partial F}{\partial u} \) is bounded in the neighborhood of face \( \frac{I}{2} \), the boundary of B, then

\[ F(u_I) - F(\beta u_{I-j} + (1 - \beta)u_{I-j+1}) = \tilde{\theta}(\Delta x) \]  

(27)

Similarly, if \( \frac{\partial F}{\partial u} \) is bounded in the neighborhood of face \( I + \frac{I}{2} \), the boundary of A, then

\[ F(u_I) - F(\alpha u_{m+1} + (1 - \alpha)u_{m+1}) = \tilde{\theta}(\Delta x) \]  

(28)

From Equations (27), (28), and (18), it is obvious that

\[ E = \left| \tilde{F}_{I+\frac{I}{2}} - \tilde{F}_{I}\right| = \tilde{\theta}(\Delta x) \]  

(29)

i.e., the conservation error is a first-order term.

To the best of our knowledge, the error estimate shown in this paper is the first quantitative measure ever given for a non-conservative data interpolation scheme for the Chimera method. We believe that the conclusion is also true for two and three-dimensions although the proof is not given here.

Remarks.
• The proof given in this section is based on the assumption that a steady-state solution always exists and can be obtained. In practice, this assumption is not guaranteed. Therefore it is possible that an instability near the overlapped region can cause the solution process to diverge.

• With a fixed-length overlapped region, the conservation error as defined by equation (10) is one order lower than the order of the conservative scheme used in the two overlapped domains. Therefore, the conservation error may remain a constant with global grid refinement if first-order conservative schemes are used in the overlapped domains. It is highly recommended that at least second-order conservative schemes should be used with the original Chimera grid scheme.

The estimate on the conservation error doesn’t address whether or not, or how the conservation error affects a numerical solution on a grid with finite resolution. This question has to be answered with numerical tests.

**Demonstration Cases**

**Turbulent Flow Over Two-Element Airfoil.**

To verify the conservation error estimate given in the last section, NASA’s OVERFLOW code (version 1.6ap)\(^{14}\) with both the original (non-conservative) and conservative Chimera (CC) was used in a grid refinement study. The so-called conservative Chimera approach was originally developed for cell-centered finite method by Wang\(^{13}\). The based idea is that overlapped grids are converted into patched grids by using a common patch boundary for cell cutting. Then fluxes (mass, momentum and energy) are conserved on the common patch boundary to ensure both local and global conservation. The conservative Chimera was implemented into OVERFLOW recently.
Refer to Reference 15 and 16 for implementation details. It was frequently reported in the literature that the original Chimera (OC) may have difficulties in handling shocks. This phenomenon can be explained with the error analysis presented in the last section. It has never been shown, however, whether the conservation error in the original Chimera affects flows without discontinuities. A grid refinement study was therefore conducted with conservative and non-conservative Chimera for a subsonic turbulent flow case using OVERFLOW. The geometry is a two-element airfoil which is shown in Figure 2. For display clarity, the blocked cells and one more layer of grid cells are not shown in this Figure. The flow conditions are: \( M_\infty = 0.4; \alpha = 2.26; \text{Re} = 9 \times 10^6 \). Three grids (coarse, medium, and fine) were generated and used in this grid refinement study. Shown in Figure 2 is the course grid \((241x81 + 121x31)\). The fine grid is twice as fine in both directions. All three grids have identical domain boundaries. In the computations with the original Chimera, it was made sure that no orphan grid points existed on any of the three grids, and that the overset grids always overlapped each other with one to two cells with approximately the same hole locations. This was done to ensure a fair comparison with the conservative Chimera in which the grids patched each other (i.e., with a zero cell overlap). To enable an objective comparison between the conservative and non-conservative Chimera, all other factors such as the position of the Chimera-hole, numerical scheme, time-marching scheme, CFL, turbulent model, etc., were the same. The “Chimera holes” for CC and OC are displayed in Figures 3 and 4, respectively. Note that the hole locations are almost identical. Both the conservative and non-conservative Chimera had no problems in converging the solutions on all three grids. The pressure and Mach contours calculated with CC and OC on the fine grid are nearly indistinguishable from each other, and therefore not shown here. The \( C_p \) profiles on the airfoil surfaces on all three grids are plotted for both CC and OC in Figure 5. A close-up view of
the Cp profiles on the flap is presented in Figure 6. It is obvious that on the fine grid, the conservative and non-conservative Chimera gave almost identical Cp profiles, verifying that the conservation error indeed diminishes with grid refinement. On the coarse and medium grids, conservative Chimera gave significantly better solution (closer to the fine grid solution) than the original Chimera. Since all other factors in the simulation were kept the same, it is the only logical conclusion that the conservation error contributed to the large difference shown between the conservative and non-conservative Chimera on the coarse and medium grids. The difference is indeed striking given the fact that no shock waves exist in the flow.

**A Steady Shock on Overlapped Grids.**

The error estimate presented in the second section is valid given the assumption that no discontinuities coincide with the overlapped interface boundaries. To test what actually happens when a shock is located at the interface boundary, the case of a steady shock wave is used with the original Chimera. The flow is assumed to travel from left to right. The conditions on the left and right of the shock wave are:

\[
\begin{align*}
\rho_L &= 1, & \rho_R &= 1.8621 \\
u_L &= 1.5, & u_R &= 0.80554 \\
p_L &= 0.71429, & p_R &= 1.7559
\end{align*}
\]

For this case, CFDRC’s CFD-FASTRAN code\(^{17}\) was used in the simulation. Since the conservative Chimera is fully conservative, it has no difficulties in capturing the discontinuities wherever they are located. With Roe’s flux difference splitting\(^{18}\), the conservative Chimera captured exactly the steady shock, while with van Leer’s flux vector splitting\(^{19}\), the conservative
Chimera captured the shock with two transition points. The results with the conservative Chimera are well known, and are therefore not shown here. Instead the computational results with the original Chimera are presented.

If the shock is located away from the overlapped interface, the steady shock is preserved exactly by Roe’s approximate Riemann solver, as shown in Figure 7. When the shock is located at the overlapped grid interface, a spurious solution was produced with Roe’s Riemann solver, as shown in Figure 8. Initially, an oscillation was generated at the overlapped boundary (also the shock location). This oscillation then propagates down stream, and a spurious constant state was developed for the entire computational domain. What is most amazing is that this spurious constant state does not disturb the steady shock wave in the other domain. In other words, Roe’s Riemann solver not only admits steady shocks and expansion shocks, but also subsonic shocks where the pre- and post-shock states are subsonic. Although this spurious solution is not the fault of the original Chimera scheme, the interpolation scheme induced this spurious solution.

Van Leer’s flux vector splitting was also used for this case with the original Chimera. If the shock wave is away from the interface, van Leer’s splitting can also preserve the shock, but with a two cell transition as shown in Figure 9. If the shock is located at the interface, oscillations first develop from the interface and then the steady shock cannot be preserved as shown in Figure 10. With both Roe and van Leer flux splitting, the spurious oscillations does not diminish with global grid refinement, indicating a source of error independent of the grid size. This case demonstrates that the non-conservative Chimera indeed may cause the solution process to diverge if a shock is located at the overlapped grid interface. With the conservative Chimera scheme, the shock wave
is resolved without any difficulty no matter where the shock wave is located because the scheme is globally as well as locally conservative. A solution similar to that shown in Figure 7 was obtained with Roe splitting regardless of the location of the shock wave. With van Leer splitting, the solution was similar to that shown in Figure 9. Therefore the solutions with the conservative Chimera are not plotted.

Conclusions

An estimate on the conservation error due to the non-conservative data interpolation scheme for overset grids is given in this paper. If there are shock waves located at the overlapped grid interface, non-conservative Chimera may have problems in capturing the shock waves even with global grid refinement. Other than that, the conservation error is a first-order term if second-order conservative schemes are employed for the Chimera grids. Therefore, in the limit of global grid refinement, valid numerical solutions should be obtained with a data interpolation scheme.

In one demonstration case, the conservation error in the original Chimera scheme was clearly shown to affect flows even without discontinuities on coarse to medium grids through a comparison with a conservative scheme with all other factors being the same. The conservative Chimera scheme gave significantly better results than the original Chimera for a smooth flow test case on the coarse and medium grids. On a very fine grid, the conservative Chimera and non-conservative Chimera indeed converged to the same solution. In realistic 3D flow simulations, the resolution of the computational grids are usually limited and are not fine enough for the non-conservative Chimera to yield the same solution as that of the conservative Chimera. Therefore, conservative Chimera may give better solutions, even for smooth flows, in these cases.
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References


Figure 1. Two Overlapping Regions A and B with Overlapped Region 0

Figure 2. The Coarse Grid for Turbulent Flow Over a Two-Element Airfoil

Figure 3. Patch Interfaces for the Conservative Chimera in the Coarse Grid

Figure 4. Holes Cut for the Original Chimera in the Coarse Grid

Figure 5. Comparison of Cp Profiles Along the Two-Element Airfoil

Figure 6. Comparison of Cp Distributions Along the Rear Airfoil

Figure 7. A Steady Shock Preserved with the Original Chimera (Shock Wave Away from the Interfaces, Roe Flux Difference Splitting)

Figure 8. A Steady Shock Not Preserved with the Original Chimera (Shock Wave at One of the Interfaces, Roe Flux Difference Splitting)

Figure 9. A Steady Shock Preserved with the Original Chimera (Shock Wave Away from the Interfaces, van Leer Flux-Vector Splitting)

Figure 10. A Steady Shock not Preserved with the Original Chimera (Shock Wave at One of the Interfaces, van Leer Flux-Vector Splitting)
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